Reedy Model Structures in Families

Edouard Balzin

Abstract

Given a family of model categories $\mathcal{E} \to \mathcal{R}$ over a Reedy category, we outline a set of conditions which lead to the existence of a Reedy model structure on the category of sections $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$. We prove that for a wide class of examples, this model structure serves as a strictification of the $(\infty, 1)$-category of sections of the higher-categorical family associated to $\mathcal{E} \to \mathcal{R}$.

Contents

Introduction

Let $\mathcal{M}$ be a model category. Given a small category $\mathcal{C}$, the associated functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ can be endowed with objectwise weak equivalences: take those natural transformations $X \to Y$ such that for each $c \in \mathcal{C}$, the value $X(c) \to Y(c)$ is a weak equivalence in $\mathcal{M}$. For a general [24] choice of $\mathcal{M}$ (assuming small (co)limits, see Definition 2.2) and $\mathcal{C}$, there is no known way to complete the class of objectwise weak equivalences on $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ into a model structure. One option to treat this issue is to assume additional requirements about $\mathcal{M}$, such as being cofibrantly generated, or further, combinatorial. Another way concerns the assumptions on $\mathcal{C}$.

The notion of a Reedy category is attributed to [25]. To fix a definition, a *Reedy category* is a small category $\mathcal{R}$ possessing two subcategories $\mathcal{R}_- \subset \mathcal{R} \supset \mathcal{R}_+$ with $\mathrm{Ob}\,\mathcal{R}_- = \mathrm{Ob}\,\mathcal{R} = \mathrm{Ob}\,\mathcal{R}_+$, and a degree function $deg : \mathcal{R} \to \mathbb{N}$ taking values in natural numbers, such that

1. the isomorphisms of $\mathcal{R}$ are identities,

2. the non-identities of $\mathcal{R}_-$ lower the value of deg, and the non-identities of $\mathcal{R}_+$ raise the value of deg,

3. any morphism $f : x \to y$ factors uniquely as $f = f^+ \circ f_-$, where $f_- \in \mathcal{R}_-$ and $f^+ \in \mathcal{R}_+$.

Many elementary diagram categories are naturally Reedy: for example, the two-arrow category $a \longleftarrow b \longrightarrow c$ appearing in the calculus of pullbacks, can be endowed with at least two Reedy structures; in the first one, both maps raise the degree, in the other, both maps lower the degree. Similarly, the category

$$[n] = 0 \longrightarrow ... \longrightarrow n$$

admits two different Reedy structures: for the first structure, $deg(0) = 0$ and all maps advance the degree, for the second, $deg(0) = n$ and all maps lower the degree. Taking all $[n]$ for $n \in \mathbb{N}$ defines a full subcategory $\Delta \subset \mathbf{Cat}$ of the category of small categories. The surjections, the injections and $deg([n]) = n$ make Δ (and Δ^{op}) into a Reedy category. Furthermore, for any category $\mathcal{C}$, denote by $\Delta\mathcal{C}$ the category of functors $\mathbf{c} : [n] \to \mathcal{C}$, with morphisms $\mathbf{c} \to \mathbf{c}'$ given by functors $[n] \to [m]$ in the overcategory $\mathbf{Cat}/\mathcal{C}$. The Reedy structure on Δ induces a Reedy structure on $\Delta\mathcal{C}$ [8, 22.10]. This example shows that Reedy categories exist in a great variety.

The classical result (see e.g. [17]) asserts that given a Reedy category $\mathcal{R}$ and a model category $\mathcal{M}$, there exists an explicitly-constructed model structure on $\mathsf{Fun}(\mathcal{R}, \mathcal{M})$, with objectwise weak equivalences. The existence of a model structure on functors from a Reedy category permits the computation of various homotopy limits, even for a general category $\mathcal{C}$: the assignment

$$(\mathbf{c} : [n] \to \mathcal{C}) \mapsto \mathbf{c}(n)$$

defines a (homotopy cofinal) functor [8, 22.11] $p_t : \Delta\mathcal{C} \to \mathcal{C}$. We can use the pull-back along p_t to embed $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ in $\mathsf{Fun}(\Delta\mathcal{C}, \mathcal{M})$, and then use the Reedy model structure on $\mathsf{Fun}(\Delta\mathcal{C}, \mathcal{M})$ for homotopy colimit computations.

Sections over a Reedy category

In this paper, we study families of model categories and their associated categories of sections. In light of the Grothendieck construction [10, 32], a natural way to treat a family of categories indexed by $\mathcal{C}$ is to consider a functor $\mathcal{E} \to \mathcal{C}$ together with additional conditions (that we survey in Section 1).

Given a family $\mathcal{E} \to \mathcal{C}$, we can associate to it the category of sections $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$, consisting of those functors $S : \mathcal{C} \to \mathcal{E}$ such that $p \circ S = id_{\mathcal{C}}$. The case of functors $\mathcal{C} \to \mathcal{M}$ can be recovered by considering the trivial family, given by projection $pr_{\mathcal{C}} : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$: the sections of $pr_{\mathcal{C}}$ are identified with $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$.

For an example of a nontrivial model-categorical family, denote by **CRing** the category of commutative rings and consider a small diagram $A : \mathcal{C} \to \mathbf{CRing}$. For a ring R, denote by $C^{\bullet}(\mathbf{Mod}_R)$ the category of chain complexes of modules over R, that can be endowed with projective model structure [17]. Define $\mathbf{Mod}_A$ to be the category consisting of pairs (c, M) where c is in $\mathcal{C}$ and M is a complex of $A(c)$-modules. A morphism $(c, M) \to (c', N)$ in $\mathcal{E}$ is given by a map $f : c \to c'$ in $\mathcal{C}$ and a chain map $M \to N|_{A(c)}$ of complexes of $A(c)$-modules, where we restrict N along the map $A(f) : A(c) \to A(c')$; by adjunction, it is the same data as a map of complexes of $A(c')$-modules $A(c') \otimes_{A(c)} M \to N$. This example of algebras and modules can be generalised [30, 15], with the abstraction being the notion of a Quillen presheaf [16, 3], which is the same data as a functor from a small category $\mathcal{C}$ to model categories and Quillen adjunctions.

When $\mathcal{C} = \Delta$, the category of sections $\mathsf{Sect}(\Delta, \mathbf{Mod}_A)$ plays an important role in the questions of cohomological descent: inside $\mathsf{Sect}(\Delta, \mathbf{Mod}_A)$, there is a full subcategory denoted $\mathsf{Sect}_{\mathbb{L}cart}(\Delta, \mathbf{Mod}_A)$ consisting of sections S that send $[n] \to [m]$ in Δ to a quasi-isomorphism $A(m) \otimes_{A(n)}^{\mathbb{L}} S(n) \to S(m)$. As explained in [30], the category $\mathsf{Sect}_{\mathbb{L}cart}(\Delta, \mathbf{Mod}_A)$ is the category of descent data in derived algebraic geometry, see also Remark 3.48.

To work with sections homotopically, one would like to, just as in the case of functors to a model category, have a model structure. We thus pose the following problem: given a family $p : \mathcal{E} \to \mathcal{R}$ over a Reedy category $\mathcal{R}$, formulate the requirements on p for the category $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ to have a model structure, that specialises to the Reedy model structure in the case of a trivial family.

Our findings are summarised by Theorem 2.11 in the main text:

Theorem 1. *Let $\mathcal{R}$ be a Reedy category and $\mathcal{E} \to \mathcal{R}$ an* admissible model semifibration. *Then the category of sections* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ *has a model structure with objectwise weak equivalences.*

Let us explain the conditions imposed on $\mathcal{E} \to \mathcal{R}$.

The functor $p : \mathcal{E} \to \mathcal{R}$ cannot be completely general: as it turns out, the appropriate condition on p is that of a *semifibration*, which is a mixture of the conditions of Grothendieck fibration and opfibration. In detail, the semifibration property consists in requiring that the restriction $\mathcal{E}|_{\mathcal{R}_-} \to \mathcal{R}_-$ is a Grothendieck prefibration; the restriction $\mathcal{E}|_{\mathcal{R}_+} \to \mathcal{R}_+$ is dually

3

assumed to be a preopfibration. Denoting $\mathcal{E}(c) := p^{-1}(c)$, these two conditions imply that for each $f_- : x \to y$ in $\mathcal{R}_-$, there is a functor $f_-^* : \mathcal{E}(y) \to \mathcal{E}(x)$, and dually for each $f^+ : x \to y$ in $\mathcal{R}_+$, there is a functor $f_!^+ : \mathcal{E}(x) \to \mathcal{E}(y)$. The final requirement of a semifibration relates these two classes of functors, see Definition 1.38 and Proposition 1.40 for detail.

Given an object $x \in \mathcal{R}$ of a Reedy category $\mathcal{R}$, denote $Lat(x) \subset \mathcal{R}/x$ the "latching category", consisting of all $\mathcal{R}_+$-maps $y \to x$ without the identity map. Similarly, the "matching category" $Mat(x) \subset x\backslash\mathcal{C}$ consists of all $\mathcal{R}_-$-maps $x \to y$ minus the identity. The semifibration property of $\mathcal{E} \to \mathcal{R}$ implies the existence of two restriction functors $L_x : \mathcal{E}|_{Lat(x)} \to \mathcal{E}(x)$ and $R_x : \mathcal{E}|_{Mat(x)} \to \mathcal{E}(x)$. Given a section $S : \mathcal{R} \to \mathcal{E}$, the latching and the matching object functors are defined as follows:

$$\mathscr{L}_x S := \varinjlim_{Lat(x)} L_x S|_{Lat(x)}, \quad \mathscr{M}_x S := \varprojlim_{Mat(x)} R_x S|_{Mat(x)}$$

(we assume that all needed (co)limits exist). Given a section $S : \mathcal{R} \to \mathcal{E}$ of the semifibration $\mathcal{E} \to \mathcal{R}$, its value $S(x)$ fits into a diagram

$$\mathscr{L}_x S \to S(x) \to \mathscr{M}_x S$$

and just as in the classical Reedy case, such diagrams for different x completely control the behaviour of S.

The semifibration $\mathcal{E} \to \mathcal{R}$ being *model* means that each fibre $\mathcal{E}(x)$ is a model category, and that we also require the transition functors $f_!^+ : \mathcal{E}(x) \to \mathcal{E}(y)$ along $\mathcal{R}_+$-maps to preserve cofibrations and trivial cofibrations (we do not require the preservation of colimits, which permits to consider such functors as tensor products, and consequently the examples of algebras and TQFT [19]). A dual condition of preserving fibrations and trivial fibrations should hold for $f_-^* : \mathcal{E}(y) \to \mathcal{E}(x)$. For each map of sections $S \to T$, there is a naturally induced diagram

$$
\begin{array}{ccc}
\mathscr{L}_x S & \longrightarrow S(x) \longrightarrow & \mathscr{M}_x S \\
\downarrow & \downarrow & \downarrow \\
\mathscr{L}_x T & \longrightarrow T(x) \longrightarrow & \mathscr{M}_x T.
\end{array}
$$

We define $S \to T$ to be a *Reedy cofibration* if for each $x \in \mathcal{R}$, the map $\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$ is a cofibration in $\mathcal{E}(x)$. Dually, $S \to T$ is a *Reedy fibration* if for each $x \in \mathcal{R}$, the map $S(x) \to \mathscr{M}_x S \prod_{\mathscr{M}_x T} T(x)$ is a fibration in $\mathcal{E}(x)$. The Reedy weak equivalences are defined objectwise.

The defined classes of maps give a model structure on $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$, provided that the model semifibration $\mathcal{E} \to \mathcal{R}$ satisfies the *admissibility* condition, Definition 2.9. The admissibility guarantees that the functor $S \mapsto \mathscr{L}_x S$ sends (trivial) Reedy cofibrations to (trivial) cofibrations, and dually for $\mathscr{M}_x$. The examples of admissible families are covered by Lemma 2.10. A Quillen presheaf is admissible, since its transition functors preserve (co)limits. A Grothendieck fibration in model categories and right derivable functors over $[n]$ is also admissible, which follows from the simple structure of the matching categories of $[n]$.

The proof of Theorem 1 is similar to the one of the classical case. The explicit description of the model structure permits to verify various properties, such as being cofibrantly generated under reasonable assumptions, Propositions 2.30 and 2.32.

Comparison with higher-categorical sections

The second part of our work establishes a relation with the higher category theory (that we model using the language of quasicategories [18, 6]). To begin, recall that given a category $\mathcal{M}$ with weak equivalences $\mathcal{W}$, its $(\infty, 1)$-localisation is an $(\infty, 1)$-functor $F : \mathcal{M} \to L_{\mathcal{W}}\mathcal{M}$, such that for any $(\infty, 1)$-category $\mathcal{Z}$, the induced $(\infty, 1)$-functor $F^* : \mathsf{Fun}(L_{\mathcal{W}}\mathcal{M}, \mathcal{Z}) \to \mathsf{Fun}(\mathcal{M}, \mathcal{Z})$ is full and faithful, and its essential image consists of $(\infty, 1)$-functors that send $\mathcal{W}$ to equivalences of $\mathcal{Z}$. For model categories, the $(\infty, 1)$-localisation enjoys various special properties [11]. In particular, Cisinski [6] has shown that given any model category $\mathcal{M}$ and a small category $\mathcal{C}$, the natural $(\infty, 1)$-functor $\mathsf{Fun}(\mathcal{C}, \mathcal{M}) \to \mathsf{Fun}(\mathcal{C}, L_{\mathcal{W}}\mathcal{M})$ induces an $(\infty, 1)$-equivalence

$$L\,\mathsf{Fun}(\mathcal{C}, \mathcal{M}) \to \mathsf{Fun}(\mathcal{C}, L_{\mathcal{W}}\mathcal{M})$$

where on the left, we localise with respect to the objectwise weak equivalences.

Thanks to the various literature on quasicategories [18, 6, 9, 11, 20] there is now a well-developed theory of Grothendieck fibrations of $(\infty, 1)$-categories. In some cases [11, 20], it is known how to $(\infty, 1)$-localise a family $\mathcal{E} \to \mathcal{R}$ to get a properly behaved higher-categorical family $L\mathcal{E} \to \mathcal{R}$. We thus ask how the localisation $L\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ of the model structure of Theorem 1 is compared with the $(\infty, 1)$-sections $\mathsf{Sect}(\mathcal{R}, L\mathcal{E})$.

We treat the comparison issue in the following generality. A functor $\mathcal{E} \to \mathcal{R}$ is a *left model Reedy fibration* if it is a Grothendieck opfibration, a Grothendieck fibration over $\mathcal{R}_-$, and is an admissible model semifibration. Localising $\mathcal{E}$ along the union $\cup_c \mathcal{W}(c)$ of the fibrewise weak equivalences yields an $(\infty, 1)$-functor $L\mathcal{E} \to \mathcal{R}$ that is a coCartesian fibration (in the sense of [18, Definition 2.4.2.1]) and a Cartesian fibration over $\mathcal{R}_-$. The natural functor $\mathcal{E} \to L\mathcal{E}$ induces the $(\infty, 1)$-functor $\mathsf{Sect}(\mathcal{R}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{R}, L\mathcal{E})$. The following result is Theorem 3.37:

Theorem 2. *Let $\mathcal{E} \to \mathcal{R}$ be a left model Reedy fibration. Then the induced infinity-functor $L\mathsf{Sect}(\mathcal{R}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{R}, L\mathcal{E})$ is an equivalence of quasicategories.*

The origins of Theorem 2 lie in the paper [16]. However, the proof of [16, Théorème 18.2] is incomplete: the authors do not provide proofs for the needed higher-categorical arguments, and the model-categorical considerations of [16, Théorème 18.2] contain a mistake (it is assumed that the latching object functor preserves fibrant objects). Nonetheless, our proof of Theorem 2 shows that the infinity-categorical Reedy induction can be carried out along the general lines of [16]. There are other comparison results [13, 28] that work under more assumptions on the family $\mathcal{E} \to \mathcal{R}$. For example, [13] bypasses the Reedy induction, yet is only valid for Quillen presheaves in the combinatorial model setting.

The example of Quillen presheaves has a particularly good strictification result, conjectured in [16] and outlined in Proposition 3.42:

Proposition 3. *Let* $\mathcal{M} \to \mathcal{C}$ *be a Quillen presheaf over a small category* $\mathcal{C}$. *Then localising* $\mathcal{M}$ *along the fibrewise weak equivalences yields an equivalence of* $(\infty, 1)$-*categories*

$$L\mathsf{Sect}(\mathcal{C}, \mathcal{M}) \xrightarrow{\sim} \mathsf{Sect}(\mathcal{C}, L\mathcal{M}).$$

where we treat $\mathsf{Sect}(\mathcal{C}, \mathcal{M})$ *as a category equipped with objectwise weak equivalences, with no model structure.*

Organisation of the paper

Section 1 covers various categorical preliminaries that are related to the semifibrations and Reedy categories. We outline the induction for sections using the notions of Noether categories, Definition 1.21. The notion of a semifibration is introduced in Definition 1.38. Section 1 includes more material than is required for Section 2, however, we decided to keep many propositions for future reference. This comment also applies to the Appendix, which gives a variation of the argument leading to Theorem 1 in a specialised setting.

Section 2 establishes the notion of a model semifibration and admissibility, Definition 2.9, and proves Theorem 1. We discuss the cofibrant generation of Reedy model structures, proving Propositions 2.30 and 2.32.

In Section 3 we address the comparison theorem. We start by recalling a result of Lurie [18, Proposition A.2.9.14] asserting the higher-categorical Reedy induction for the functors to a quasicategory. We then generalise it to the case of a coCartesian fibration $\mathcal{E} \to \mathcal{R}$ of quasicategories over a Reedy category that is also Cartesian over $\mathcal{R}_-$ and is suitably bicomplete, Proposition 3.17. After studying some aspects of relative categories in families, we prove a model-categorical counterpart, Proposition 3.40, of Proposition 3.17. Both these propositions lead to the proof of Theorem 2. We conclude Section 3 by discussing the strictification of Quillen presheaves.

Acknowledgements

1 Semifibrations

1.1 Cartesian arrows, prefibrations, sections

Let $p : \mathcal{E} \to \mathcal{C}$ be a functor. For $c \in \mathcal{C}$, denote by $\mathcal{E}(c)$ the fibre category $p^{-1}(c)$ over c. It thus consists of all $X \in \mathcal{E}$ with $p(X) = c$ and all the maps $X \to X'$ with $p(X \to X') = id_c$.

Definition 1.1. A morphism $\alpha : X \to Y$ of $\mathcal{E}$

- is p-*cartesian*, or simply cartesian, if for any other map $\beta : X' \to Y$ with $p(\beta) = p(\alpha)$ there exists a unique morphism $\gamma : X' \to X$ in $\mathcal{E}(p(X))$ which factors β as $\alpha \circ \gamma$.

- is p-*opcartesian*, or simply opcartesian, if for any other map $\delta : X \to Y'$ with $p(\delta) = p(\alpha)$ there exists a unique morphism $\eta : Y \to Y'$ in $\mathcal{E}(p(Y))$ which factors δ as $\eta \circ \alpha$.

A p-cartesian or p-opcartesian morphism $\alpha : X \to Y$ is covering the morphism $f : c \to c'$ iff $p(\alpha) = f$.

In our definition of cartesian and opcartesian morphisms, we are faithful to the original terminology of [10]. Today, a different definition of (op)cartesian maps is presented in many sources [32, 19], with the definition of [10] referred to as "locally cartesian" morphism.

Definition 1.2. A functor $p : \mathcal{E} \to \mathcal{C}$ is a

- prefibration iff for any $f : x \to y$ of $\mathcal{C}$ and $Y \in \mathcal{E}(y)$ there exists a cartesian morphism $\alpha : X \to Y$ covering f, that is, $p(\alpha) = f$.

- preopfibration iff for any $f : x \to y$ of $\mathcal{C}$ and $X \in \mathcal{E}(x)$ there exists a cartesian morphism $\delta : X \to Z$ covering f, that is, $p(\alpha) = f$.

Lemma 1.3. *If $p : \mathcal{E} \to \mathcal{C}$ is a prefibration, then $p^{\mathsf{op}} : \mathcal{E}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ is a preopfibration.* $\qquad\square$

Notation 1.4. If $p : \mathcal{E} \to \mathcal{C}$ is a prefibration, $f : x \to y$ is a morphism and $Y \in \mathcal{E}(y)$, we shall usually denote a chosen cartesian lift by $f^*Y \to Y$. The same applies when p is a preopfibration, where for $X \in \mathcal{E}(x)$, we denote by $X \to f_!X$ the chosen opcartesian lift.

Definition 1.5. A prefibration or preopfibration $q : \mathcal{E} \to \mathcal{C}$ is small if both $\mathcal{C}$ and $\mathcal{E}$ are small categories. A prefibration or preopfibration q is *discrete* if for each $c \in \mathcal{C}$, the category $\mathcal{E}(c)$ has no non-identity maps (in other words, it is isomorphic to a set).

Lemma 1.6. *Let $p : \mathcal{E} \to \mathcal{C}$ be a discrete prefibration. Then the composition of cartesian morphisms of $\mathcal{E}$ is cartesian. The dual is true for preopfibrations.*

Proof. Clear, due to the lack of fibre maps. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In general, not any pre(op)fibration has the property described in the previous lemma. Those which have it, are called Grothendieck (op)fibrations.

Definition 1.7. A prefibration $p : \mathcal{E} \to \mathcal{C}$ is, furthermore, a *Grothendieck fibration* iff the composition of cartesian maps is cartesian. The definition for *Grothendieck opfibrations* is dual.

Discrete pre(op)fibrations are thus automatically (op)fibrations. The examples of non-discrete fibrations are, however, abundant.

Remark 1.8. It is not necessarily the case that the category $\mathcal{E}$ is "bigger" than $\mathcal{C}$. For example, the functor $\mathcal{C} \to \mathcal{C} \coprod \mathcal{D}$ is a fibration and an opfibration.

Remark 1.9. In what follows, (op)fibrations will be considered as special cases of pre(op)fibrations, with additional remarks where necessary. Otherwise, any definition or a result given for a pre(op)fibration implies the same for an (op)fibration.

Construction 1.10. Given a functor E from $\mathcal{C}$ to categories, we produce an opfibration, which we denote $\int E \to \mathcal{C}$ and call the *Grothendieck construction* [32] of E. An object of $\int E$ is a pair (c, X) of $c \in \mathcal{C}$ and $X \in E(c)$, and a morphism $(c, X) \to (c', X')$ consists of $f : c \to c'$ together with a map $\alpha : E(f)(X) \to X'$ in $E(c')$.

Dually, for a contravariant category-valued functor F defined on $\mathcal{C}$, its Grothendieck construction is a fibration $\int F \to \mathcal{C}$ with same pairs (c, Y) serving as objects, but with maps given by pairs of $f : c \to c'$ and $\beta : Y \to F(f)Y'$ in $F(c)$.

Grothendieck construction motivates the following perspective. Consider a prefibration $p : \mathcal{E} \to \mathcal{C}$. Let $f : c \to c'$ be a morphism in $\mathcal{C}$ and $Y \in \mathcal{E}(c')$. Choose a cartesian morphism $\alpha : f^*Y \to Y$ covering f. This specifies an object $f^*Y \in \mathcal{E}(c)$. By the universal property of cartesian maps, the assignment $Y \mapsto f^*Y$ defines a functor $f^* : \mathcal{E}(c') \to \mathcal{E}(c)$, which is called transition functor along f. Due to the universal property of cartesian arrows, for each composable pair f, g, there exists a 'coherence' natural transformation $f^* \circ g^* \to (g \circ f)^*$, which is an isomorphism if p is a Grothendieck fibration. For any composable triple of arrows f, g, h, any choice of coherence morphisms leads to the following commutative diagram:

$$
\begin{array}{ccc}
f^*g^*h^* & \longrightarrow & (gf)^*h^* \\
\downarrow & & \downarrow \\
f^*(hg)^* & \longrightarrow & (hgf)^*.
\end{array}
\qquad (1.1)
$$

For a preopfibration, the whole picture is dual. In the literature (see [10] and [32] for the case of Grothendieck fibrations), such choice of an assignment $f \mapsto f^*$ together with coherence isomorphisms is called a cleavage. One may thus wonder if there is a way to obtain (lax) category-valued functors from (pre)fibrations.

Definition 1.11. Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{E}' \to \mathcal{C}$ be two functors.

- A morphism of p and q is a functor $F : \mathcal{E} \to \mathcal{E}'$ commuting with the functors to $\mathcal{C}$, that is, $q \circ F = p$.

- A section of p is a functor $S : \mathcal{C} \to \mathcal{E}$ such that $p \circ S = id_{\mathcal{C}}$. In other words, it is a morphism from $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ to $p : \mathcal{E} \to \mathcal{C}$.

Given two morphisms $F, F' : \mathcal{E} \to \mathcal{E}'$, a morphism between them is a natural transformation $\alpha : F \to F'$ such that for each X in the domain $\mathcal{E}$, α_X projects to $id_{p(X)}$.

We denote by $\mathsf{Lax}(\mathcal{E}, \mathcal{E}')$ the category of morphisms between p and q, with the functors to $\mathcal{C}$ being implicit. By $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) = \mathsf{Lax}(\mathcal{C}, \mathcal{E})$ we denote the category of sections of p.

Definition 1.12. Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{E}' \to \mathcal{C}$ be two prefibrations (respectively preopfibrations). A morphism $F : \mathcal{E} \to \mathcal{E}'$ is called a *cartesian morphism* if it takes (op)cartesian morphisms of $\mathcal{E}$ to (op)cartesian morphisms of $\mathcal{E}'$.

We denote by $\mathsf{Cart}(\mathcal{E}, \mathcal{E}')$ the full subcategory of $\mathsf{Lax}(\mathcal{E}, \mathcal{E}')$ consisting of cartesian morphisms.

Construction 1.13. Take a fibration $p : \mathcal{E} \to \mathcal{C}$, and for each $c \in \mathcal{C}$, denote by $\mathcal{C}/c$ the category of objects over c [22]. The forgetful functor $\mathcal{C}/c \to \mathcal{C}$ is an fibration. Then the assignment $c \mapsto \mathsf{Cart}(\mathcal{C}/c, \mathcal{E})$ defines a contravariant category-valued functor on $\mathcal{C}$. When $\mathcal{C}$ is small, this construction is inverse up to an equivalence [32] to (Grothendieck) Construction 1.10.

If p is only a prefibration, the assignment $c \mapsto E(c) = \mathsf{Cart}(\mathcal{C}/c, \mathcal{E})$ defines a lax contravariant functor from $\mathcal{C}$ to categories. Indeed, for each $f : c \to c'$, we get a functor $f^* : E(c') \to E(c)$, and as before, one can witness the existence of natural transformations $f^* g^* \to (gf)^*$ and of the diagram like (*1.1*).

The cited result [32] implies that any fibration (and, similarly, an opfibration) $p : \mathcal{E} \to \mathcal{C}$ can be, up to an equivalence, replaced by an fibration $\tilde{p} : \tilde{\mathcal{E}} \to \mathcal{C}$, for which the assignment $c \mapsto \mathcal{E}(c)$ can be made into a strict functor by a choice of transition functors along maps in $\mathcal{C}$. We call the fibrations (similarly, fibrations) with the latter property *strictly cleavable*.

Definition 1.14. A functor $p : \mathcal{E} \to \mathcal{C}$ is an *isofibration* if for any isomorphism $f : c \xrightarrow{\sim} d$ of $\mathcal{C}$ and an object Y with $p(Y) = d$ there exists an isomorphism $\alpha : X \xrightarrow{\sim} Y$ with $p\alpha = f$.

A Grothendieck (op)fibration is automatically an isofibration, but a pre(op)fibration is not. In particular, in an arbitrary prefibration, a cartesian lift of an isomorphism is not necessarily an isomorphism.

Convention 1.15. From now on, any prefibration or preopfibration we consider is assumed to be also an isofibration. For an isofibration $p : \mathcal{E} \to \mathcal{C}$ and $c \in \mathcal{C}$, the notation $\mathcal{E}(c)$ will denote

$p^{-1}(c)$, the strict categorical fibre of p over c. Note that in this case, the strict fibre is equivalent to the *essential fibre* of p over c: the objects of the latter are pairs of $d \in \mathcal{D}$ and $\alpha : p(d) \cong c$ in $\mathcal{C}$, and morphisms $(d, \alpha) \to (d', \beta)$ are given by $f : d \to d'$ such that $\beta p(f) = \alpha$. In particular, $p(f)$ is an isomorphism.

Example 1.16. Let $L : \int \mathcal{E} \to \int \mathcal{E}'$ be a morphism between two Grothendieck constructions of covariant functors $\mathcal{E}, \mathcal{E}' : \mathcal{C} \to \mathbf{Cat}$. For each $c \in \mathcal{C}$, L specifies a functor $L_c : \mathcal{E}(c) \to \mathcal{E}'(c)$. For each morphism $f : c \to c'$, we get a 2-square

$$
\begin{array}{ccc}
\mathcal{E}(c) & \xrightarrow{L_c} & \mathcal{E}'(c) \\
{\scriptstyle \mathcal{E}(f)}\downarrow & \overset{L_f}{\Leftarrow} & \downarrow{\scriptstyle \mathcal{E}'(f)} \\
\mathcal{E}(c') & \xrightarrow[L_{c'}]{} & \mathcal{E}'(c').
\end{array}
$$

The natural transformation appears because the image under L of an opcartesian map $X \to \mathcal{E}(f)X$ ($X \in \mathcal{E}(c)$) may not be opcartesian. Factoring $LX \to L\mathcal{E}(f)X$,

$$LX \to \mathcal{E}'(f)LX \to L\mathcal{E}(f)X,$$

gives $\mathcal{E}'(f)LX \to L\mathcal{E}(f)X$; for each $X \in \mathcal{E}(c)$, all such maps assemble into L_f. For two composable arrows $f : c \to c'$, $g : c' \to c''$, there is a pasting property relating L_f, L_g and L_{gf}: the pasting of this diagram

$$
\begin{array}{ccccc}
\mathcal{E}(c) & \xrightarrow{\mathcal{E}(f)} & \mathcal{E}(c') & \xrightarrow{\mathcal{E}(g)} & \mathcal{E}(c'') \\
{\scriptstyle L_c}\downarrow & \overset{L_f}{\Downarrow} & \downarrow{\scriptstyle L_{c'}} & \overset{L_g}{\Downarrow} & \downarrow{\scriptstyle L_{c''}} \\
\mathcal{E}'(c) & \xrightarrow[\mathcal{E}'(f)]{} & \mathcal{E}'(c') & \xrightarrow[\mathcal{E}'(g)]{} & \mathcal{E}'(c'')
\end{array}
$$

equals L_{gf}.

For fibrations, there is a difference on the level of 2-diagrams. Consider $\mathcal{F}, \mathcal{F}' : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and take a lax morphism $M : \int \mathcal{F} \to \int \mathcal{F}'$ of fibrations over $\mathcal{C}$. For $f : c \to c'$, we then obtain a diagram

$$
\begin{array}{ccc}
\mathcal{F}(c) & \xrightarrow{M_c} & \mathcal{F}'(c) \\
{\scriptstyle \mathcal{F}(f)}\uparrow & \overset{M_f}{\Rightarrow} & \uparrow{\scriptstyle \mathcal{F}'(f)} \\
\mathcal{F}(c') & \xrightarrow[M_{c'}]{} & \mathcal{F}'(c')
\end{array}
$$

with M_f given by arrows of the form $M\mathcal{F}(f)Y \to \mathcal{F}'(f)MY$.

1.2 Limits and adjunctions

Consider a Grothendieck prefibration $\mathcal{E} \to \mathcal{C}$ over a base $\mathcal{C}$. Let us study the question when the category of sections $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ admits limits or colimits. As a related question, given a pullback square of fibrations

$$
\begin{array}{ccc}
F^*\mathcal{E} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{D} & \xrightarrow{\;F\;} & \mathcal{C}
\end{array}
$$

we ask if the natural restriction functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ admits an adjoint.

Basic results

Definition 1.17. A functor $\mathcal{E} \to \mathcal{C}$ is *fibrewise-complete* if every fibre $\mathcal{E}(c)$ is complete. Likewise, $\mathcal{E} \to \mathcal{C}$ is *fibrewise-cocomplete* if every fibre $\mathcal{E}(c)$ is cocomplete.

A fibration, opfibration, prefibration or preopfibration is fibrewise complete or cocomplete if it is true on the level of the underlying functor.

Proposition 1.18. *Let $\mathcal{E} \to \mathcal{C}$ be a prefibration which is fibrewise cocomplete. Then the category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ is cocomplete, with colimits calculated fibrewise. The dual result concerns limits in the category of sections of a complete preopfibration.*

Proof. Let $S_\bullet : I \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ be a diagram of sections,

$$
(i, c) \in I \times \mathcal{C} \;\mapsto\; S_i(c) \in \mathcal{E}(c).
$$

We then define $(\varinjlim_I S_\bullet)(c) = \varinjlim_I S_i(c)$, that is, the colimit of $S_\bullet(c) : I \to \mathcal{E}(c)$ in the fibre $\mathcal{E}(c)$. Take a morphism $f : c \to d$, it then suffices to construct

$$
(\varinjlim_I S_\bullet)(c) \to f^*(\varinjlim_I S_\bullet)(d) \tag{1.2}
$$

for some choice of a cartesian morphism $f^*(\varinjlim_I S_\bullet)(d) \to (\varinjlim_I S_\bullet)(d)$. If we choose cartesian morphisms for each $i \in I$, obtaining the diagram

$$
f^*S_\bullet(d) : I \to \mathcal{E}(c), \;\; i \mapsto f^*S_i(d),
$$

then we have the canonical morphism

$$
\varinjlim_I f^*S_\bullet(d) \to f^*(\varinjlim_I S_\bullet(d))
$$

induced by the colimit property. Combining it with the map $\varinjlim_I S_\bullet(c) \to \varinjlim_I f^*S_\bullet(d)$ induced by the section structure of $S_\bullet$, we get the map (1.2). One can check that the induced maps are compatible with the composition of morphisms in $\mathcal{C}$ in a suitable way. We leave it to the reader: everything follows, in essence, from the universality of maps from a colimit.

Let $X \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ be a section, and denote by $c^*X : I \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ the constant diagram valued at X. Given a map $S_\bullet \to c^*X$, we want to construct an adjoint map $\varinjlim_I S_\bullet \to X$. First, we can construct, fibre by fibre, the maps

$$\varinjlim_I S_\bullet(c) \to X(c).$$

For a morphism $f : c \to d$, we can then draw the diagram

$$
\begin{array}{ccccc}
\varinjlim_I S_\bullet(c) & \longrightarrow & \varinjlim_I f^*S_\bullet(d) & \longrightarrow & f^* \varinjlim_I S_\bullet(d) \\
\downarrow & & \downarrow & & \downarrow \\
X(c) & \longrightarrow & f^*X(d) & \overset{=}{\longrightarrow} & f^*X(d)
\end{array}
$$

The left square commutes because $S_\bullet \to c^*X$ is a morphism of sections, the right square commutes due to the universal property of colimits. We thus see that the family of fibrewise maps gives a morphism of sections $\varinjlim_I S_\bullet \to X$. The verification in the other direction is similar. $\qquad\square$

Given a pullback square of prefibrations,

$$
\begin{array}{ccc}
F^*\mathcal{E} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{D} & \overset{F}{\longrightarrow} & \mathcal{C},
\end{array}
$$

the assignment $S \mapsto S \circ F$ defines a functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, \mathcal{E})$. One would tentatively write, then, the left adjoint $F_!$ to F^* as a certain colimit over the comma category F/c. However, the fibration structure does not permit for sensible formulas to appear. What remains true is the following:

Proposition 1.19. *Let $\mathcal{E} \to \mathcal{C}$ be a fibrewise-cocomplete prefibration, and*

$$
\begin{array}{ccc}
F^*\mathcal{E} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{D} & \overset{F}{\longrightarrow} & \mathcal{C}
\end{array}
$$

be a pullback square. Assume that $F : \mathcal{D} \to \mathcal{C}$ is an opfibration. Then $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ admits a left adjoint $F_!$, which can be calculated as

$$F_!T(c) = \varinjlim_{\mathcal{D}(c)} T|_{\mathcal{D}(c)}.$$

Proof. Straightforward and similar to Proposition 1.18. Note that $F : \mathcal{D} \to \mathcal{C}$ being an opfibration implies that the natural functor $\mathcal{D}(c) \to F/c$ admits a left adjoint and is hence cofinal. $\qquad\square$

Locally Noether categories

In what follows, we shall use the words "sequence" and "chain" interchangeably.

Definition 1.20. Let $\mathcal{C}$ be a category, and $c \in \mathcal{C}$ be an object. We say that c is k-*bounded from the right* for some $k \in \mathbb{N}$ if any sequence of n morphisms starting with c,

$$c \longrightarrow c_1 \longrightarrow ... \longrightarrow c_n$$

contains at least $n - k$ isomorphisms so long as $n > k$. Dually, c is k-bounded from the left if any sequence of n morphisms ending with c,

$$c_n \longrightarrow c_{n-1} \longrightarrow ... \longrightarrow c_1 \longrightarrow c$$

contains at least $n - k$ isomorphisms so long as $n > k$

We shall often say "bounded" without being precise about the direction when it leads to no confusion.

Definition 1.21. A category $\mathcal{C}$ is called *locally Noetherian*, or simply a *Noether category* if for each object $c \in \mathcal{C}$ there exists a number k, such that c is k-bounded from the right.

Dually, a category $\mathcal{C}$ is called *locally Artinian*, or simply an *Artin category* if for each object $c \in \mathcal{C}$ there exists a number k, such that c is k-bounded from the left.

Remark 1.22. Evidently, if $\mathcal{C}$ is a Noether category, then $\mathcal{C}^{\mathsf{op}}$ is an Artin category. We shall henceforth stick with the Noether case in our considerations, but all the results can of course be dualised for the Artin case.

For a Noether category $\mathcal{C}$ and $c \in \mathcal{C}$, denote by $|c| \geq 0$ the minimal such k so that c is k-bounded from the right.

Lemma 1.23. *For $c, c' \in \mathcal{C}$, if $|c| < |c'|$, then $\mathcal{C}(c, c') = \emptyset$. If $|c| = |c'|$ and there is a map $c \to c'$, then it is an isomorphism. In particular, any endomorphism of c is an isomorphism.*

Proof. Let $c' \to c'_1 \to ... \to c'_{|c'|}$ be a chain starting with c of length $|c'|$ such that no map in the sequence is an isomorphism. If there is a map $c \to c'$ in $\mathcal{C}$, composing with it would yield a sequence of maps of length $|c'| + 1$ starting from c.

Thus, if $|c| < |c'|$, we have a sequence of non-invertible maps of length $|c'| + 1$ starting from c, out of which at least $|c'|$ maps are non-invertible, and this is impossible. If $|c| = |c'|$, having a map $c \to c'$ becomes only possible if it is an isomorphism. $\qquad\square$

We thus have a degree function $c \mapsto |c|$, which can be considered as a contravariant functor $|-| : \mathcal{C}^{\mathsf{op}} \to \mathbb{N}$ to the category $\mathbb{N}$ of natural numbers and unique morphisms in positive direction.

Notation 1.24. For a Noether category $\mathcal{C}$, denote by $\mathcal{C}_n$ the subcategory of objects c such that $|c| \leq n$. There is an induced filtration $\mathcal{C}_0 \subset \mathcal{C}_1 \subset ... \subset \mathcal{C}_n \subset ... \subset \mathcal{C}$. Denote also by $\mathcal{G}_n$ the subcategory of $\mathcal{C}$ consisting of c with $|c| = n$. Lemma 1.23 implies that $\mathcal{G}_n$ is a groupoid.

Let $\mathcal{E} \to \mathcal{C}$ be a prefibration. For $x \in \mathcal{C}$, if $\mathcal{D}$ is a subcategory $x \backslash \mathcal{C}$, then the prefibration property implies the existence of a functor $Res_x : \mathcal{E}|_{\mathcal{D}} \to \mathcal{E}(x)$. An object $Y \in \mathcal{E}|_{\mathcal{D}}$ living over $f : x \to y$ of $\mathcal{D}$ is sent to f^*Y where $f^*Y \to Y$ is a cartesian map. The choice of Res_x is unique up to a unique isomorphism.

Let S be a section over $\mathcal{C}_{n-1}$. Consider the limit $\varprojlim_{c \backslash \mathcal{C}_{n-1}} Res_c S$ where $c \in \mathcal{G}_n$. Since the maps $c \to c'$ are isomorphisms for $|c| = |c'|$, we naturally have $\mathcal{E}(c) \cong \mathcal{E}(c')$ (see Convention 1.15) and we get a canonically determined map $\varprojlim_{c \backslash \mathcal{C}_{n-1}} Res_c S \to \varprojlim_{c' \backslash \mathcal{C}_{n-1}} Res_{c'} S$.

Definition 1.25. Let $\mathcal{E} \to \mathcal{C}$ be a prefibration over a Noether category $\mathcal{C}$ and $S \in \mathsf{Sect}(\mathcal{C}_{n-1}, \mathcal{E})$. The *$n$-th matching system of S*, denoted $\mathscr{M}_n S$, is the section

$$\mathscr{M}_n S : \mathcal{G}_n \to \mathcal{E}|_{\mathcal{G}_n}, \quad c \mapsto \varprojlim_{c \backslash \mathcal{C}_{n-1}} Res_c S \in \mathcal{E}(c)$$

of the prefibration $\mathcal{E} \to \mathcal{G}_n$, assuming that all the necessary limits exist.

The assignment $S \mapsto \mathscr{M}_n S$ defines a functor $\mathscr{M}_n : \mathsf{Sect}(\mathcal{C}_{n-1}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{G}_n, \mathcal{E})$.

Proposition 1.26. *There is a comma square*

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{C}_n, \mathcal{E}) & \longrightarrow & \mathsf{Sect}(\mathcal{G}_n, \mathcal{E}) \\
\downarrow & \Leftarrow & \downarrow = \\
\mathsf{Sect}(\mathcal{C}_{n-1}, \mathcal{E}) & \underset{\mathscr{M}_n}{\longrightarrow} & \mathsf{Sect}(\mathcal{G}_n, \mathcal{E})
\end{array}
$$

making $\mathsf{Sect}(\mathcal{C}_n, \mathcal{E})$ into the comma category $\mathsf{Sect}(\mathcal{G}_n, \mathcal{E})/\mathscr{M}_n$. In other words, the assignment

$$Y \in \mathsf{Sect}(\mathcal{C}_n, \mathcal{E}) \quad \mapsto \quad (Y|_{\mathcal{C}_{n-1}}, Y|_{\mathcal{G}_n}, Y|_{\mathcal{G}_n} \to \mathscr{M}_n Y|_{\mathcal{C}_{n-1}}) \in \mathsf{Sect}(\mathcal{G}_n, \mathcal{E})/\mathscr{M}_n$$

is an equivalence of categories.

Proof. Assume that we are given a section S on $\mathcal{C}_{n-1}$ and a map $X \to \mathscr{M}_n S$ of sections in $\mathsf{Sect}(\mathcal{G}_n, \mathcal{E})$. We show how to construct a new section $\tilde{S} : \mathcal{C}_n \to \mathcal{E}$. For an object $c \in \mathcal{C}_n$ of $|c| = n$, there are two kinds of maps: $c \to c'$ with $|c'| = n$ and $c \to c''$ with $|c''| < n$. The first ones are isomorphisms of $\mathcal{G}_n$ and are included in X as part of the data. The map $X \to \mathscr{M}_n S$ then provides us with morphisms $X(c) \to S(c'')$ in a manner compatible with $\mathcal{G}_n$. $\qquad \square$

Let I be a small category and denote by $X_\bullet :\in \mathsf{Sect}(\mathcal{R}, \mathcal{E})^I$ a diagram of sections,

$$(x, i) \quad \mapsto \quad X_i(x).$$

If the fibre $\mathcal{E}(x)$ admits limits, we may compute the limit of the functor $i \mapsto X_i(x)$, which we denote $\varprojlim_I (X_\bullet(x))$. We would now like to conclude if the limit of $X_\bullet$, denoted $\varprojlim_I X_\bullet$, exists globally in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$.

Proposition 1.27. *Let $\mathcal{C}$ be a Noether category and $\mathcal{E} \to \mathcal{C}$ a Grothendieck prefibration with complete fibres. Then the category of sections $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ admits limits, and moreover, for each $X_\bullet \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})^I$ and an object x with $|x| = n$, there is the following pullback square:*

$$
\begin{array}{ccc}
(\varprojlim_I X_\bullet)(y) & \longrightarrow & \varprojlim_I(X_\bullet(y)) \\
\downarrow & & \downarrow \\
\mathscr{M}_n(\varprojlim_I X_\bullet)(y) & \longrightarrow & \varprojlim_I(\mathscr{M}_n X_\bullet)(y).
\end{array}
\tag{1.3}
$$

where $\mathscr{M}_n X_\bullet : \mathcal{G}_n \times I \to \mathcal{E}$ is the functor $(y, i) \mapsto (\mathscr{M}_n X_i)(y)$.

Proof. For each x with $|x| = 0$ we define $(\varprojlim_I X_\bullet)(x) = \varprojlim_I(X_\bullet(x))$, that is we take the limit in the corresponding fibre $\mathcal{E}(x)$. Since there are no maps out of objects of degree zero, and $\mathcal{E}(x) \cong \mathcal{E}(x')$ for $x \cong x'$, we get a well-defined section $\mathcal{C}_0 \to \mathcal{E}$.

Having specified $(\varprojlim_I X_\bullet)$ on $\mathcal{C}_{n-1}$, the diagram (1.3) tells us precisely how to define the value $(\varprojlim_I X_\bullet)(y)$ for $y \in \mathcal{G}_n$. The right vertical arrow exists as a limit of the natural map $X_\bullet(y) \to (\mathscr{M}_n X_\bullet)(y)$. The bottom horizontal arrow exists because, by induction, there are natural maps $(\varprojlim_I X_\bullet)(x) \to X_i(x)$ for $x \in \mathcal{C}_{n-1}$. These maps induce $\mathscr{M}_n(\varprojlim_I X_\bullet)(y) \to (\mathscr{M}_n X_i)(y)$ and then, consequently, we get a map to $\varprojlim(\mathscr{M}_n X_\bullet)(y)$.

To verify that the constructed section $Y = \varprojlim_I X_\bullet$ is the limit in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$, proceed by induction (which is trivial in degree zero) and consider a map $c^* Z \to X_\bullet$, where $c^* Z$ is the constant I-section valued at $Z : \mathcal{C}_n \to \mathcal{E}$. For each y with $|y| = n$, we then get the following diagram:

$$
\begin{array}{ccc}
Z(y) & \longrightarrow & \varprojlim_I(X_\bullet(y)) \\
\downarrow & & \downarrow \\
\mathscr{M}_n Z(y) \longrightarrow \mathscr{M}_n Y(y) & \longrightarrow & \varprojlim_I(\mathscr{M}_n X_\bullet)(y)
\end{array}
$$

which is commutative because it is simply a factoring of the commutative diagram

$$
\begin{array}{ccc}
Z(y) & \longrightarrow & \varprojlim_I(X_\bullet(y)) \\
\downarrow & & \downarrow \\
\mathscr{M}_n Z(y) & \longrightarrow & \varprojlim_I(\mathscr{M}_n X_\bullet)(y)
\end{array}
$$

where the factoring $\mathscr{M}_n Z(y) \to \mathscr{M}_n Y(y) \to \varprojlim_I(\mathscr{M}_n X_\bullet)(y)$ exists due to the limit property of Y on $\mathcal{C}_{n-1}$. We thus get the commutative square

$$
\begin{array}{ccc}
Z(y) & \longrightarrow & \varprojlim_I(X_\bullet(y)) \\
\downarrow & & \downarrow \\
\mathscr{M}_n Y(y) & \longrightarrow & \varprojlim_I(\mathscr{M}_n X_\bullet)(y)
\end{array}
$$

which, by the pullback property of the diagram (1.3), supplies us with $Z(y) \to Y(y)$, as desired. $\square$

Proposition 1.26 can be relativised. Recall the following notions:

Definition 1.28. A functor $F : \mathcal{D} \to \mathcal{C}$ is

- An open immersion if it is full, faithful, injective on objects, and for each $f : c \to F(d)$ of $\mathcal{C}$ there exists a (unique) map $\tilde{f} : d' \to d$ in $\mathcal{D}$ covering f.

- An closed immersion if it is full, faithful, injective on objects, and for each $f : F(d) \to c$ of $\mathcal{C}$ there exists a (unique) map $\tilde{f} : d \to d'$ in $\mathcal{D}$ covering f.

Recall that, for $c \in \mathcal{C}$, a cosieve is a subcategory $S \subset c\backslash\mathcal{C}$ closed under postcomposition: $f : c \to c' \in S$ implies that gf is in S for any $g : c' \to c''$ of $\mathcal{C}$.

Lemma 1.29. *For a functor $F : \mathcal{D} \to \mathcal{C}$ injective on objects, the following are equivalent*

- *F is a closed immersion,*

- *F is a faithful isofibration (Definition 1.14), and for each $d \in \mathcal{D}$, the essential image of $d\backslash\mathcal{D}$ in $F(d)\backslash\mathcal{C}$ is a cosieve.*

- *F is a fully faithful Grothendieck opfibration with discrete fibres.*

The dual is true for an open immersion.

Proof. Clear. $\square$

In particular, let $c \in \mathcal{C}$ be an object not contained in the image of F. Then $\mathcal{C}(F(d), c) = \emptyset$ for any $d \in \mathcal{D}$. Thus, at most, there are only morphisms going out of c to $\mathcal{D}$.

Let $\mathcal{C}$ be a Noether category and $F : \mathcal{D} \to \mathcal{C}$ a closed immersion. In what follows, we identify $\mathcal{D}$, which is also a Noether category, with its image in $\mathcal{C}$.

Notation 1.30. Define $\mathcal{D}_n$ to be the subcategory consisting of $\mathcal{D}$ and all the objects $c \in \mathcal{C}$ not belonging to $\mathcal{D}$ with $|c| \leq n$. Denote by $F_n : \mathcal{D} \to \mathcal{D}_n$ the inclusion functor. There is also an inclusion $\mathcal{D}_n \to \mathcal{C}$ which we leave unnamed. Finally, denote by $\mathcal{G}_n$ the subcategory of $\mathcal{D}_n$ consisting of those objects c which do not belong to $\mathcal{D}_{n-1}$.

For an object $c \in \mathcal{C}$ (usually assumed to be outside in $\mathcal{G}_n$) we can define the category $c\backslash\mathcal{D}_{n-1}$ as the usual comma category for the inclusion $\mathcal{D}_{n-1} \to \mathcal{C}$: its objects are maps $c \to d$ in $\mathcal{C}$, where d belongs to $\mathcal{D}_{n-1}$.

As usual for comma categories and prefibrations, we get the restriction functor $Res_c : \mathcal{E}|_{c\backslash\mathcal{D}_{n-1}} \to \mathcal{E}(c)$.

Proposition 1.31. *Let $F : \mathcal{D} \to \mathcal{C}$ be a closed immersion of Noether categories and $\mathcal{E} \to \mathcal{C}$ be a prefibration with complete fibres. Then any section $X \in \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ admits a right Kan extension $Ran_F X \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ which restricts to right Kan extensions $Ran_{F_n} X \in \mathsf{Sect}(\mathcal{D}_n, \mathcal{E})$ of X along $F_n : \mathcal{D} \to \mathcal{D}_n$. Moreover, $F^* Ran_F X \cong X$ and for any $x \in \mathcal{G}_n$,*

$$(Ran_{F_n} X)(x) = \varprojlim_{x \backslash \mathcal{D}_{n-1}} Res_x \circ Ran_{F_{n-1}} X \tag{1.4}$$

where we implicitly restrict $Ran_{F_{n-1}} X$ to $x \backslash \mathcal{D}_{n-1}$ along the evident projection.

Proof. We construct $Ran_{F_n} X$ for each value of n by induction. For $n = 0$, the only objects of $x \in \mathcal{D}_0$ which are not in $\mathcal{D}$ are those which admit no non-invertible maps out of themselves, since $|x| = 0$. We thus pose $(Ran_{F_0} X)(x)$ to be a terminal object of $\mathcal{E}(x)$. The formula (1.4) then explains how to carry on the induction: for $x, y \in \mathcal{D}_n$ which are not in $\mathcal{D}_{n-1}$, the maps $x \to y$, if exist, are invertible, and the construction of $(Ran_{F_n} X)(x) \to (Ran_{F_n} X)(y)$ is thus as trivial as in Proposition 1.26. Finally, each object (or a morphism, or a composition of morphisms) of $\mathcal{C}$ belongs to some $\mathcal{G}_n$, which permits us to define $Ran_F X$ on the whole of $\mathcal{C}$.

Since F is a closed immersion, $F^* Ran_F X$ is verified, using (1.4), to be isomorphic to X. Let $T \in \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ be a section and assume we have a map $\alpha : F^* T \to X$. We would like now to obtain a (canonical) morphism $\beta : T \to Ran_F X$. Assume by induction (which is again trivially initiated for objects of zero degree) that we obtained this map for all $c \in \mathcal{D}_{n-1}$ in a compatible fashion. Let now x be an object of $\mathcal{G}_n$. There is a diagram in $\mathcal{E}(x)$ of the form

$$T(x) \to \varprojlim_{x \backslash \mathcal{D}_{n-1}} Res_x \circ T \to \varprojlim_{x \backslash \mathcal{D}_{n-1}} Res_x \circ Ran_{F_{n-1}} X = Ran_{F_n} X(x)$$

where, if needed, both T and $Ran_{F_{n-1}} X$ are restricted to $x \backslash \mathcal{D}_{n-1}$. The first map exists due to the section structure of T, the second map is given by the inductive assumption, and together they provide $T(x) \to Ran_{F_n} X(x) = Ran_F X(x)$. The described assignment is a bijection, as verified quite easily by applying F^*. $\square$

The assignment $X \mapsto Ran_F X$ thus defines a fully faithful functor $F_* : \mathsf{Sect}(\mathcal{D}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ right adjoint to F^*.

Consider a closed immersion $F : \mathcal{C}' \to \mathcal{C}$ and an object $c \in \mathcal{C}$. One can form the following pullback square in **Cat**

$$
\begin{array}{ccc}
c \backslash \mathcal{C}' & \xrightarrow{\pi'} & \mathcal{C}' \\
F_c \downarrow & & \downarrow F \\
c \backslash \mathcal{C} & \xrightarrow{\pi} & \mathcal{C}
\end{array}
$$

with $c \backslash \mathcal{C}'$ coinciding with the usual comma category $c \backslash F$. Moreover, one can verify that each category in this diagram is Noether, with all functors preserving the degrees and the vertical ones, F and F_c, being closed immersions (the functors π and π', while being discrete Grothendieck fibrations, are merely faithful).

17

If we are given a fibrewise complete prefibration over $\mathcal{C}$, then there is the following induced 2-diagram

$$
\begin{array}{ccc}
\mathsf{Sect}(c\backslash\mathcal{C}', \mathcal{E}) & \xleftarrow{\pi'^*} & \mathsf{Sect}(\mathcal{C}', \mathcal{E}) \\
{\scriptstyle F_{c,*}}\Big\downarrow & \Leftarrow & \Big\downarrow{\scriptstyle F_*} \\
\mathsf{Sect}(c\backslash\mathcal{C}, \mathcal{E}) & \xleftarrow[\pi^*]{} & \mathsf{Sect}(\mathcal{C}, \mathcal{E}).
\end{array}
$$

Proposition 1.32. *In the diagram above, the map $\pi^* F_* \to F_{c,*}\pi'^*$ is an isomorphism.*

We prove it by induction, forming, for each $c \in \mathcal{C}$, denote by $\mathcal{C}'_n$ and $(c\backslash\mathcal{C}')_n$ the induction categories as in Notation 1.30, with $\pi_n : (c\backslash\mathcal{C}')_n \to \mathcal{C}'_n$. being the projection functor. One can see that, moreover, $(c\backslash\mathcal{C}')_n \cong c\backslash\mathcal{C}'_n$. Then Proposition 1.32 will follow from

Proposition 1.33. *Let $F : \mathcal{C}' \to \mathcal{C}$ be a closed immersion of Noether categories and $\mathcal{E} \to \mathcal{C}$ be a prefibration with complete fibres. Then for each n the 2-square*

$$
\begin{array}{ccc}
\mathsf{Sect}(c\backslash\mathcal{C}', \mathcal{E}) & \xleftarrow{\pi'^*} & \mathsf{Sect}(\mathcal{C}', \mathcal{E}) \\
{\scriptstyle Ran_{F_{c,n}}}\Big\downarrow & \Leftarrow & \Big\downarrow{\scriptstyle Ran_{F_n}} \\
\mathsf{Sect}(c\backslash\mathcal{C}'_n, \mathcal{E}) & \xleftarrow[\pi_n^*]{} & \mathsf{Sect}(\mathcal{C}'_n, \mathcal{E}).
\end{array}
$$

commutes up to an isomorphism.

Proof. We shall proceed by induction on n. For $n = 0$, the extension to objects of degree zero outside of $\mathcal{C}'$ or $c\backslash\mathcal{C}'$ is given by terminal objects, hence the isomorphism is trivial. Take now an object of $c\backslash\mathcal{C}'_n$, represented by a map $c \to d$ with d outside of t i$\mathcal{C}'$ and the degree of $|c \to d| = |d|$ equal to n. We can then write that

$$
\pi_n^* Ran_{F_n} X(c \to d) = Ran_{F_n} X(d) = \varprojlim_{d\backslash\mathcal{C}'_{n-1}} Res_d \pi_{n-1}^* Ran_{F_{n-1}} X
$$

with π_{n-1} here being the functor $d\backslash\mathcal{C}'_{n-1} \to \mathcal{C}'_{n-1}$, and also that

$$
F_{c,*}\pi'^* X(c \to d) = \varprojlim_{(c\to d)\backslash(c\backslash C'_{n-1})} Res_{c\to d} Ran_{F_{c,n-1}} \pi'^* X \cong \varprojlim_{d\backslash C'_{n-1}} Res_d Ran_{F_{c,n-1}} \pi'^* X
$$

where in the middle term one more restriction is implicit. By induction,

$$
\pi_{n-1}^* Ran_{F_{n-1}} X \to Ran_{F_{c,n-1}} \pi'^* X
$$

is an isomorphism, which induces the isomorphism between the two limit expressions above. $\square$

18

1.3 Factorisation systems and semifibrations

Definition 1.34. A *factorisation system* on a category $\mathcal{C}$ consists of a pair of subcategories $\mathscr{L}, \mathscr{R} \subset \mathcal{C}$ containing all isomorphisms of $\mathcal{C}$, such that any morphism $f : c \to c'$ in $\mathcal{C}$ can be decomposed as

$$f : c \xrightarrow{\ l\ } c'' \xrightarrow{\ r\ } c' \tag{1.5}$$

with $l \in \operatorname{Mor}\mathscr{L}$ and $r \in \operatorname{Mor}\mathscr{R}$. This factorisation must be moreover unique up to unique isomorphism.

In this work, a *factorisation category* will denote a triple $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ of a category together with a factorisation system $(\mathscr{L}, \mathscr{R})$.

When clear, we shall simply refer to a factorisation category $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ as $\mathcal{C}$. Due to the isomorphism condition $\mathscr{L}$ and $\mathscr{R}$ contain all the objects of $\mathcal{C}$. We shall often refer to $\mathscr{L}$ as the left class of maps, and to $\mathscr{R}$ as the right class of maps.

Definition 1.35. A strict *factorisation functor* $F : (\mathcal{C}', \mathscr{L}', \mathscr{R}') \to (\mathcal{C}, \mathscr{L}, \mathscr{R})$ is a functor $\mathcal{C}' \to \mathcal{C}$ such that $F(\mathscr{L}') \subset \mathscr{L}$ and $F(\mathscr{R}') \subset \mathscr{R}$. We shall occasionally denote by $F_L : \mathscr{L}' \to \mathscr{L}$ and $F_R : \mathscr{R}' \to \mathscr{R}$ the induced functors.

An important class of factorisation categories is given by Reedy categories. To repeat,

Definition 1.36. A *Reedy category* $\mathcal{R}$ is a factorisation category $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$ together with a degree function $deg : \mathcal{R} \to \mathbb{N}$ taking values in natural numbers, such that

1. the isomorphisms of $\mathcal{R}$ are identities,

2. the non-identities of $\mathcal{R}_-$ lower the value of deg,

3. the non-identities of $\mathcal{R}_+$ raise the value of deg,

It is implied that $\mathcal{R}_-$ is locally Noether and $\mathcal{R}_+$ is locally Artin. For the literature concerning Reedy categories, see [18, 12, 17, 26]. We assume the degree function to be taking values in natural numbers. While this suffices for most practical examples, our choice excludes from consideration the case $\mathcal{R} = \beta$ for an arbitrary ordinal β. The latter has some importance for the theory-building of Section 3, but is a relatively mild case and will be treated by hand once needed. Henceforth, we shall also be implicit about the degree function in our notation.

The following definition concerns the way factorisation functors interact with the factorisations (1.5).

Definition 1.37. Let $F : (\mathcal{C}', \mathscr{L}', \mathscr{R}') \to (\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation functor. We say that F is *right-closed* if for any $\mathcal{C}$-map of the form $c \to F(c')$, the $(\mathscr{L}, \mathscr{R})$-factorisation of this map can be chosen as

$$c \xrightarrow{\ l\ } F(c'') \xrightarrow{F(r)} F(c')$$

with $r : c'' \to c'$ belonging to $\mathscr{R}'$. Dually, F is *left-closed*, if for any $\mathcal{C}$-map of the form $F(c') \to c$, the $(\mathscr{L}, \mathscr{R})$-factorisation of this map can be chosen as

$$F(c') \xrightarrow{F(l)} F(c'') \xrightarrow{r} c$$

with $l : c' \to c''$ belonging to $\mathscr{L}'$.

Semifibrations

Definition 1.38. Let $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation category. A functor $p : \mathcal{E} \to \mathcal{C}$ is called a *semifibration* over $\mathcal{C}$ if it is an isofibration and the following conditions are satisfied.

1. For any $l : c \to c'$ in $\mathscr{L}$ and Y with $p(Y) = c'$ there exists a cartesian (Definition 1.1) lift $\lambda : Y' \to Y$ of l.

2. For any $r : x \to y$ in $\mathscr{R}$ and X with $p(X) = x$ there exists an opcartesian lift $\rho : X \to X'$ of r.

3. For any $\alpha : X \to Y$ of $\mathcal{E}$ such that $p(\alpha)$ decomposes as

$$p(X) \xrightarrow{r} c \xrightarrow{l} p(Y)$$

   with $r \in \mathscr{R}$ and $l \in \mathscr{L}$, we require that α factors as

$$\alpha : X \xrightarrow{\rho} X' \xrightarrow{\varphi} Y' \xrightarrow{\lambda} Y \tag{1.6}$$

   with $\rho : X \to X'$, being an opcartesian morphism over r, $\lambda : Y' \to Y$ being a cartesian morphism over l, and $p(\varphi) = id_c$.

Lemma 1.39. *The third condition of Definition 1.38 is equivalent to the following: for any $\alpha : X \to Y$ of $\mathcal{E}$ such that $p(\alpha)$ decomposes as*

$$p(X) \xrightarrow{r} c \xrightarrow{l} p(Y)$$

with $r \in \mathscr{R}$ and $l \in \mathscr{L}$, we require that

$$\alpha : X \xrightarrow{\rho} X' \xrightarrow{\varphi} Y' \xrightarrow{\lambda} Y$$

with $\rho : X \to X'$, being a morphism over l, $\lambda : Y' \to Y$ a morphism over r, and $p(\varphi) = id_c$.

Proof. Follows from the universality of op(cartesian) arrows. $\square$

Given a semifibration $p : \mathcal{E} \to \mathcal{C}$, If $f : c \to c'$ is a map in $\mathcal{L}$, then there is a functor $f^* : \mathcal{E}(c') \to \mathcal{E}(c)$ naturally induced by cartesian lifts. If $g : x \to y$ is a map in $\mathcal{R}$, we equally have $g_! : \mathcal{E}(x) \to \mathcal{E}(y)$ induced by opcartesian lifts.

Proposition 1.40. *Let $p : \mathcal{E} \to \mathcal{C}$ be a semifibration over $(\mathcal{C}, \mathscr{L}, \mathscr{R})$. Then*

1. The factorisation (1.6) is natural and unique up to unique isomorphism,

2. Let

$$
\begin{array}{ccc}
x & \xrightarrow{\ f\ } & y \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
z & \xrightarrow{\ k\ } & t
\end{array}
$$

be a commutative diagram with $f, k \in \mathscr{L}$ and $g, h \in \mathscr{R}$. We then have a two-square

$$
\begin{array}{ccc}
\mathcal{E}(x) & \xleftarrow{\ f^*\ } & \mathcal{E}(y) \\
{\scriptstyle g_!}\big\downarrow & \Rightarrow & \big\downarrow{\scriptstyle h_!} \\
\mathcal{E}(z) & \xleftarrow{\ k^*\ } & \mathcal{E}(t)
\end{array}
$$

with the natural transformation $g_! f^* \to k^* h_!$ induced canonically.

Proof. The first assertion is clear given the universal properties of cartesian and opcartesian morphisms.

For the second assertion, take $Y \in \mathcal{E}(y)$. Then we get the diagram in $\mathcal{E}$

$$
\begin{array}{ccc}
Y \xleftarrow{\ cart\ } f^*Y \xrightarrow{\ ocart\ } g_! f^*Y \\[1em]
\quad\searrow{\scriptstyle ocart} \\[0.5em]
\qquad\quad h_! Y \xleftarrow{\ cart\ } k^* h_! Y
\end{array}
$$

with maps labeled as *cart* being cartesian from the fibration structure over $\mathscr{L}$, and likewise *ocart* being opcartesian from the opfibration structure over $\mathscr{R}$. Then, since $hf = kg$, the composition $f^*Y \to Y \to h_! Y$ lies over $x \xrightarrow{g} z \xrightarrow{k} t$, and so, by (3) of Definition 1.38, it can be decomposed as

$$
f^*Y \to g_! f^*Y \to k^* h_! Y \to h_! Y
$$

and we get a morphism $g_! f^*Y \to k^* h_! Y$ as desired. $\qquad\square$

Since $\mathcal{C}$ is a factorisation category, any morphism $x \xrightarrow{g} z \xrightarrow{k} t$ with g in $\mathscr{R}$ and k in $\mathscr{L}$ can be completed to a diagram

$$
\begin{array}{ccc}
x & \xrightarrow{\ f\ } & y \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
z & \xrightarrow{\ k\ } & t
\end{array}
$$

as in Proposition 1.40 above. So the base-change property for the transition functors can be obtained if one assumes one of the following.

Lemma 1.41. *Let* $(\mathcal{C}, \mathcal{L}, \mathcal{R})$ *be a factorisation category and* $\mathcal{E} \to \mathcal{C}$ *be an*

- *either a fibration over* $\mathcal{C}$ *which is a preopfibration over* $\mathcal{R}$,

- *or an opfibration over* $\mathcal{C}$ *which is a prefibration over* $\mathcal{L}$,

then $\mathcal{E} \to \mathcal{C}$ *is a semifibration.*

Proof. In the first case, for the diagram

$$Y \xleftarrow{cart} f^*Y \xrightarrow{ocart} g_! f^*Y$$
$$\downarrow{ocart}$$
$$h_! Y \xleftarrow{cart} k^* h_! Y$$

as before we get that the composition $f^*Y \to Y \to h_! Y$ factors through the cartesian map $k^* h_! Y \to h_! Y$ (as implied by the stronger universal property of cartesian maps in this case [32]), so we get a map $f^*Y \to k^* h_! Y$. This map in turn is factored by the opcartesian map $f^*Y \to g_! f^*Y$, and we obtain the Y-part $g_! f^*Y \to k^* h_! Y$ of the base-change natural transformation. It can then be used to construct the factorisation of Definition 1.38. The second case is dual. $\square$

A more general statement in this direction is:

Lemma 1.42. *Let* $\mathcal{E} \to \mathcal{C}$ *be a prefibration over a factorisation category* $(\mathcal{C}, \mathcal{L}, \mathcal{R})$, *such that the restriction* $\mathcal{E}|_{\mathcal{R}} \to \mathcal{R}$ *is also a preopfibration, and such that the composition of cartesian lifts covering* $x \xrightarrow{r} z \xrightarrow{l} y$ *(with r in $\mathcal{R}$ and l in $\mathcal{L}$) is cartesian. Then* $\mathcal{E} \to \mathcal{C}$ *is a semifibration over* $(\mathcal{C}, \mathcal{L}, \mathcal{R})$.

Proof. In the proof of Lemma 1.41, we need the strong cartesian universal property exactly for arrows covering compositions like $x \xrightarrow{r} z \xrightarrow{l} y$. $\square$

1.4 Limits and adjoints in categories of sections

In a moment, we shall prove the following:

Proposition 1.43. *Let* $(\mathcal{C}, \mathcal{L}, \mathcal{R})$ *be a factorisation category and* $\mathcal{E} \to \mathcal{C}$ *be a semifibration with fibres which are complete and admit arbitrary coproducts. Assume that the category* $\mathsf{Sect}(\mathcal{L}, \mathcal{E}|_{\mathcal{L}})$ *has limits. Then so does the category* $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$. *Moreover, the restriction functor* $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{L}, \mathcal{E})$ *preserves limits.*

Dually, if $\mathcal{E} \to \mathcal{C}$ has cocomplete fibres and fibrewise products, and $\mathsf{Sect}(\mathscr{R}, \mathcal{E}|_{\mathscr{R}})$ admits colimits, then so does the category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$, and the restriction functor $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{R}, \mathcal{E})$ preserves colimits.

This can be interpreted as saying that in order to calculate limits in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$, it is sufficient to do so in $\mathsf{Sect}(\mathcal{L}, \mathcal{E})$. From now on, we shall concentrate on the limit part, the colimit part being dual.

Lemma 1.44. *Let $c \in \mathcal{C}$ and consider the undercategory $c \backslash \mathscr{L}$. Then the functor $u_c^* : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(c \backslash \mathscr{L}, \mathcal{E})$, which is induced along the natural forgetful functor $u_c : c \backslash \mathscr{L} \to \mathscr{L}$, preserves limits.*

Proof. The functor u_c^* admits a left adjoint

$$u_!^c : \mathsf{Sect}(c \backslash \mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}, \mathcal{E})$$

given by the formula $(u_!^c X)(c') = \coprod_{\mathscr{L}(c,c')} X(c')$. $\square$

For any object $c \in \mathcal{C}$, the semifibration structure provides us with the restriction functor

$$Res_c : \mathcal{E}|_{c \backslash \mathcal{L}} \to \mathcal{E}(c).$$

Proof of Proposition 1.43. Let $X_\bullet : I \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ be a diagram,

$$i \in I \ \mapsto \ (\ c \mapsto X_i(c) \ ),$$

and we would like to construct its limit $Y = \varprojlim_I X_\bullet \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$. We write the following expression

$$Y(c) = \varprojlim X_\bullet(c) = \varprojlim_{c \backslash \mathscr{L}} Res_c(\varprojlim_I^{c \backslash \mathscr{L}} X_\bullet|_{c \backslash \mathscr{L}})$$

where $\varprojlim_I^{c \backslash \mathscr{L}} X_\bullet|_{c \backslash \mathscr{L}}$ is the limit of $X_\bullet|_{c \backslash \mathscr{L}}$ taken in $\mathsf{Sect}(c \backslash \mathscr{L}, \mathcal{E})$, and we shall henceforth drop the restriction notation for $X_\bullet$.

Because the category $c \backslash \mathscr{L}$ has an initial object,

$$\varprojlim_{c \backslash \mathscr{L}} Res_c(\varprojlim_I^{c \backslash \mathscr{L}} X_\bullet) \cong (\varprojlim^{c \backslash \mathscr{L}} X_\bullet)(c \xrightarrow{id} c) \cong (\varprojlim^{\mathscr{L}} X_\bullet)(c),$$

so, thanks to Lemma 1.44, our formula is just another way to write the limit in $\mathsf{Sect}(\mathscr{L}, \mathcal{E})$.

Suppose $r : c \to d$ is a $\mathscr{R}$-map. We then need to construct $Y(r) : Y(c) \to Y(d)$. The semifibration structure implies the necessity to construct an $\mathcal{E}(d)$-map $r_! Y(c) \to Y(d)$ for some opcartesian map $Y(c) \to r_! Y(c)$. We note that for each $\mathscr{L}$-morphism $l : d \to d'$ the factorisation system of $\mathcal{C}$ implies the existence of a unique diagram

$$
\begin{array}{ccc}
c & \xrightarrow{\ r\ } & d \\
{\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle l} \\
c' & \xrightarrow{\ t\ } & d'
\end{array}
\tag{1.7}
$$

with vertical arrows in $\mathscr{L}$ and horizontal arrows in $\mathscr{R}$. In terms of undercategories, we can say that there is an induced functor

$$F : d\backslash\mathscr{L} \to c\backslash\mathscr{L}, \quad (l : d \to d') \mapsto (k : c \to c').$$

As usual, given any functor $G : c\backslash\mathscr{L} \to \mathcal{M}$ we have a natural map between limits $\varprojlim_{c\backslash\mathscr{L}} G \to \varprojlim_{d\backslash\mathscr{L}} F^*G$, provided they exist. Thus, we see that in order to construct a map f_1 in

$$r_! \varprojlim\nolimits_{c\backslash\mathscr{L}} Res_c(\varprojlim\nolimits_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_1} \varprojlim\nolimits_{d\backslash\mathscr{L}} Res_d(\varprojlim\nolimits_I^{d\backslash\mathscr{L}} X_\bullet)$$

we can attempt instead to construct another map f_2 in

$$r_! \varprojlim\nolimits_{d\backslash\mathscr{L}} F^*Res_c(\varprojlim\nolimits_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_2} \varprojlim\nolimits_{d\backslash\mathscr{L}} Res_d(\varprojlim\nolimits_I^{d\backslash\mathscr{L}} X_\bullet).$$

In turn, due to the universal property of limits, we may instead try to find a map f_3 in

$$\varprojlim\nolimits_{d\backslash\mathscr{L}} r_! F^*Res_c(\varprojlim\nolimits_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_3} \varprojlim\nolimits_{d\backslash\mathscr{L}} Res_d(\varprojlim\nolimits_I^{d\backslash\mathscr{L}} X_\bullet).$$

We can now leave out $\varprojlim_{d\backslash\mathscr{L}}$ and attempt to construct instead the morphism f_4 of functors

$$r_! F^*Res_c(\varprojlim\nolimits_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_4} Res_d(\varprojlim\nolimits_I^{d\backslash\mathscr{L}} X_\bullet).$$

Using the notation of Diagram (*1.7*), on $l : d \to d'$, the map f_4 would yield

$$r_! k^*(\varprojlim\nolimits_I^{c\backslash\mathscr{L}} X_\bullet)(c \xrightarrow{k} c') \xrightarrow{f_4(l)} l^*(\varprojlim\nolimits_I^{d\backslash\mathscr{L}} X_\bullet)(d \xrightarrow{l} d').$$

Remembering the base-change (Proposition 1.40) morphism $r_! k^* \to l^* t_!$, and the equalities

$$(\varprojlim\nolimits_I^{c\backslash\mathscr{L}} X_\bullet)(c \xrightarrow{k} c') = (\varprojlim\nolimits_I^{\mathscr{L}} X_\bullet)(c')$$

of Lemma 1.44 and the like for d, d', we see that instead of f_4 we may construct maps

$$l^* t_!(\varprojlim\nolimits_I^{\mathscr{L}} X_\bullet)(c') \xrightarrow{f_5(l)} l^*(\varprojlim\nolimits_I^{\mathscr{L}} X_\bullet)(d')$$

or even simpler, $t_!(\varprojlim_I^{\mathscr{L}} X_\bullet)(c') \to (\varprojlim_I^{\mathscr{L}} X_\bullet)(d')$. Examining $t_!(\varprojlim_I^{\mathscr{L}} X_\bullet)(c')$, we witness, naturally, that there are maps

$$t_!(\varprojlim\nolimits_I^{\mathscr{L}} X_\bullet)(c') \to t_! X_i(c') \to X_i(d')$$

with first arrow being a $t_!$ of the limit projection, and the second given by the section structure of X_i. We assemble these maps together to get $f_5(l)$ for each $l : d \to d'$, and in turn, f_4, f_3, f_2 and f_1.

This defines $Y(r) : \varprojlim X_\bullet(c) \to \varprojlim X_\bullet(d)$ for $\mathscr{R}$-maps of $\mathcal{C}$. The factorisation structure on $\mathcal{C}$ and a tedious verification (which goes through given all the maps in the reasoning above

are canonical in one way or another) then permits to see that $c \mapsto Y(c)$ is indeed a section of $\mathcal{E} \to \mathcal{C}$ that has the required universal property. $\qquad\square$

Recall that F is a right-closed factorisation functor (Definition 1.37) if for any $c \to F(c')$ there is a factorisation

$$c \xrightarrow{\ l\ } F(c'') \xrightarrow{F(r)} F(c')$$

with $r : c'' \to c'$ belonging to $\mathscr{R}' \subset \mathcal{C}'$. This implies that for each map $r : c_1 \to c_2$ of $\mathscr{R}$ we have the following diagram

$$
\begin{array}{ccc}
c_1 \backslash \mathscr{L}' & \xrightarrow{\ F_{c_1}\ } & c_1 \backslash \mathscr{L} \\
{\scriptstyle r_{\mathscr{L}'}} \big\uparrow & & \big\uparrow {\scriptstyle r_{\mathscr{L}}} \\
c_2 \backslash \mathscr{L}' & \xrightarrow{\ F_{c_2}\ } & c_2 \backslash \mathscr{L}
\end{array}
$$

with functors $r_{\mathscr{L}'}, r_{\mathscr{L}}$ given by factoring the morphisms. One has to be careful about the pullbacks of $\mathcal{E} \to \mathcal{C}$ to this diagram. If we denote by $\pi_1 : c_1 \backslash \mathscr{L} \to \mathcal{C}$, $\pi_2 : c_2 \backslash \mathscr{L} \to \mathcal{C}$ the evident projections, then the factorisations

$$
\begin{array}{ccc}
c_1 & \xrightarrow{\ r\ } & c_2 \\
{\scriptstyle k} \big\downarrow & & \big\downarrow {\scriptstyle l} \\
d_1 & \xrightarrow[\ t\ ]{} & d_2
\end{array}
\tag{1.8}
$$

which define $r_{\mathscr{L}}$ as the assignment $l \mapsto k$, imply that there is a natural transformation $\tau : \pi_1 r_{\mathscr{L}} \to \pi_2$ with components, given by maps like t in the diagram above, lying in $\mathscr{R}$.

Lemma 1.45. *Let $p : \mathcal{E} \to \mathcal{C}$ be a semifibration over $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ and $F, G : \mathcal{D} \to \mathcal{C}$ be two functors taking values in $\mathscr{L}$, and $\tau : F \to G$ be a natural transformations with components in $\mathscr{R}$. Then*

1. *both $F^*\mathcal{E} \to \mathcal{D}$ and $G^*\mathcal{E} \to \mathcal{D}$ are prefibrations,*

2. *the assignment $(X, d, F(d) = p(X)) \mapsto (\tau(d)_! X, d)$ has the property that $p(\tau(d)_! X) = G(d)$ and defines a (lax) morphism of fibrations $\tau_! : F^*\mathcal{E} \to G^*\mathcal{E}$ over $\mathcal{D}$,*

3. *there is an induced functor $\tau_! : \mathsf{Sect}(\mathcal{D}, F^*\mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, G^*\mathcal{E})$ on the categories of sections. Moreover, for each $X \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$, there is a natural (in X) map $\tau_! F^* X \to G^* X$.*

4. *Let $H : \mathcal{D}' \to \mathcal{D}$ be a functor, and assume that there are right adjoints,*

$$H_F^* : \mathsf{Sect}(\mathcal{D}, F^*\mathcal{E}) \leftrightarrows \mathsf{Sect}(\mathcal{D}', F^*\mathcal{E}) : H_*^F,$$

$$H_G^* : \mathsf{Sect}(\mathcal{D}, G^*\mathcal{E}) \leftrightarrows \mathsf{Sect}(\mathcal{D}', G^*\mathcal{E}) : H_*^G$$

   for the restriction functors H_F^*, H_G^*. Then there is a natural map

$$\tau_! H_*^F \longrightarrow H_*^G \tau_!'$$

   where $\tau_!' : \mathsf{Sect}(\mathcal{D}', H^*F^*\mathcal{E}) \to \mathsf{Sect}(\mathcal{D}', H^*G^*\mathcal{E})$ is the functor induced as in previous paragraph.

Proof. The first statement is clear. For the second we are left to prove that the assignment $X \mapsto \tau(d)_! X$ is indeed a morphism of prefibrations. For a map $f : d \to d'$, we can draw the following square

$$
\begin{array}{ccc}
Fd' & \xrightarrow{\tau(d')} & Gd' \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\
Fd & \xrightarrow{\tau(d)} & Gd
\end{array}
\qquad (1.9)
$$

Using the fibrewise-cartesian factoring on $F^*\mathcal{E}$, it remains to see what happens to the cartesian maps $Ff^*Y \to Y$, $p(Y) = Fd'$. We observe that the base-change for the diagram above implies the existence of the map

$$
\tau(d)_! F f^* Y \longrightarrow G f^* \tau(d')_! Y.
$$

Choosing (or, rather, remembering) a cartesian map $G f^* \tau(d')_! Y \to \tau(d')_! Y$, we get the composition

$$
\tau(d)_! F f^* Y \longrightarrow G f^* \tau(d')_! Y \to G f^* \tau(d')_! Y \to \tau(d')_! Y
$$

needed for constructing the functor $\tau_! : F^*\mathcal{E} \to G^*\mathcal{E}$.

The functor $\tau_!$ of the third statement is simply induced by the post-composition with the functor $\tau_!$ of the second statement. The existence of the natural family of maps $\tau_! F^* X \to G^* X$ happens for the following reason: on an object $d \in \mathcal{D}$, the map $\tau(d)_! X(F(d)) \to X(G(d))$ is supplied by the section structure of X along the $\mathscr{R}$-map $\tau(d) : F(d) \to G(d)$.

For the fourth statement, consider the diagram

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E}) & \xrightarrow{H_F^*} & \mathsf{Sect}(\mathcal{D}', F^*\mathcal{E}) \\
{\scriptstyle \tau_!}\downarrow & & \downarrow{\scriptstyle \tau_!'} \\
\mathsf{Sect}(\mathcal{D}, G^*\mathcal{E}) & \xrightarrow{H_G^*} & \mathsf{Sect}(\mathcal{D}', G^*\mathcal{E})
\end{array}
$$

and observe by explicit check that it commutes up to an isomorphism. Hence the sought-after map

$$
\tau_! H_*^F \longrightarrow H_*^G \tau_!'
$$

is given by the usual base-change argument. $\qquad\square$

Remark 1.46. The functor $\tau_! : F^*\mathcal{E} \to G^*\mathcal{E}$ takes a cartesian maps to cartesian whenever the base-change map for (*1.9*) is an isomorphism.

We would now like to prove a statement about adjoints similar to Proposition 1.43. Namely, given a semifibration $\mathcal{E} \to \mathcal{C}$ and a right-closed functor $F : \mathcal{D} \to \mathcal{C}$, we would like to deduce the existence of a right adjoint to the pullback functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ from assuming the existence of one for $F_{\mathscr{L}}^* : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}', \mathcal{E})$. However, we shall need to require some additional properties, which will make harmless the passage to comma categories.

Definition 1.47. In the situation above, we say that pull-back $F^*_{\mathscr{L}}$ admits a *pointwise right adjoint* if

1. the functor $F^*_{\mathscr{L}} : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}', \mathcal{E})$ admits a right adjoint $F_{\mathscr{L},*}$,

2. for each $c \in \mathscr{L}$, the pull-back $F^*_c : \mathsf{Sect}(c \backslash \mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(c \backslash \mathscr{L}', \mathcal{E})$ along the induced functor $F_c : c \backslash \mathscr{L}' \to c \backslash \mathscr{L}$, admits a right adjoint $F_{c,*}$ and moreover the natural base-change map $\pi^* F_{\mathscr{L},*} \to F_{c,*} \pi'^*$ arising from the square

$$
\begin{array}{ccc}
\mathscr{L}' & \xrightarrow{\ F_{\mathscr{L}}\ } & \mathscr{L} \\
{\scriptstyle \pi'} \uparrow & & \uparrow {\scriptstyle \pi} \\
c \backslash \mathscr{L}' & \xrightarrow{\ F_c\ } & c \backslash \mathscr{L}
\end{array}
$$

is an isomorphism.

In other words, this means that $F_{c,*} \pi'^* X$ can be computed as $F_{\mathscr{L},*} X$ and then restricted again to the comma category.

Proposition 1.48. *Let $F : \mathcal{C}' \to \mathcal{C}$ be a right-closed factorisation functor, and $\mathcal{E} \to \mathcal{C}$ a fibrewise complete semifibration over $\mathcal{C}$. Assume that the functor $F^*_{\mathscr{L}} : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}', \mathcal{E})$ admits a pointwise right adjoint $F_{\mathscr{L},*}$ in the sense of Definition 1.47. Then the functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ admits a right adjoint F_* such that the induced 2-diagram*

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{D}, \mathcal{E}) & \xrightarrow{\ F_*\ } & \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \\
\downarrow & \Rightarrow & \downarrow \\
\mathsf{Sect}(\mathscr{L}', \mathcal{E}) & \xrightarrow{\ F_{\mathscr{L},*}\ } & \mathsf{Sect}(\mathscr{L}, \mathcal{E}),
\end{array}
$$

(with vertical arrows given by restrictions), is in fact commutative up to an isomorphism.

We can thus make conclusions about F_* by passing to the left categories and using the functor $F_{\mathscr{L},*}$.

Proof. We shall proceed in a manner similar to Proposition 1.43. For $c \in \mathcal{C}$ and $X \in \mathsf{Sect}(\mathcal{C}', \mathcal{E})$, put

$$
Y(c) := F_* X(c) = \varprojlim_{c \backslash \mathscr{L}} Res_c F_{c,*}(X|_{c \backslash \mathscr{L}'})
$$

where $F_c : c \backslash \mathscr{L}' \to c \backslash \mathscr{L}$ is the functor induced from F. Indeed, $Y(c) \cong F_{\mathscr{L}*} X(c)$, but we will need such a presentation for Y for the proof to work.

Assume given a map $r : c_1 \to c_2$. We need to construct

$$
r_! \varprojlim_{c_1 \backslash \mathscr{L}} Res_{c_1} F_{c_1,*}(X|_{c_1 \backslash \mathscr{L}'}) \xrightarrow{\ f_1\ } \varprojlim_{c_2 \backslash \mathscr{L}} Res_{c_2} F_{c_2,*}(X|_{c_2 \backslash \mathscr{L}'})
$$

Since F is right-closed, we have the following diagram

$$
\begin{array}{ccc}
c_1\backslash\mathscr{L}' & \xrightarrow{F_{c_1}} & c_1\backslash\mathscr{L} \\
{\scriptstyle r_{\mathscr{L}'}}\big\uparrow & & \big\uparrow{\scriptstyle r_{\mathscr{L}}} \\
c_2\backslash\mathscr{L}' & \xrightarrow{F_{c_2}} & c_2\backslash\mathscr{L}
\end{array}
$$

with functors $r_{\mathscr{L}'}, r_{\mathscr{L}}$ given by factoring the morphisms. One has to be careful about the pullbacks of $\mathcal{E} \to \mathcal{C}$ to this diagram. If we denote by $\pi_1 : c_1\backslash\mathscr{L} \to \mathcal{C}$, $\pi_2 : c_2\backslash\mathscr{L} \to \mathcal{C}$ the evident projections, then the factorisations

$$
\begin{array}{ccc}
c_1 & \xrightarrow{\ r\ } & c_2 \\
{\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle l} \\
d_1 & \xrightarrow{\ t\ } & d_2
\end{array}
\qquad\qquad (1.10)
$$

imply that there is a natural transformation $\tau : \pi_1 r_{\mathscr{L}} \to \pi_2$ with components, given by maps like t in the diagram above, lying in $\mathscr{R}$.

We can thus attempt instead to construct another map f_2 in

$$
r_! \varprojlim_{c_2\backslash\mathscr{L}} r_{\mathscr{L}}^* Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_2} \varprojlim_{c_2\backslash\mathscr{L}} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'}).
$$

In turn, due to the universal property of limits, we may instead try to find a map f_3 in

$$
\varprojlim_{c_2\backslash\mathscr{L}} r_! r_{\mathscr{L}}^* Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_3} \varprojlim_{c_2\backslash\mathscr{L}} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'}).
$$

We can now leave out $\varprojlim_{c_2\backslash\mathscr{L}}$ and construct instead the morphism f_4 of functors

$$
r_! r_{\mathscr{L}}^* Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_4} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'}).
$$

Using the notation of the diagram (1.10) coming from the factorisation on $\mathcal{C}$, the map f_4 would yield

$$
r_! k^* F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_1 \xrightarrow{k} d_1) \xrightarrow{f_4(l)} l^* F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})(c_2 \xrightarrow{l} d_2).
$$

Remembering the base-change morphism $r_! k^* \to l^* t_!$, we see that instead of f_4 we may construct maps

$$
t_! F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_1 \xrightarrow{k} d_1) \xrightarrow{f_5(l)} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})(c_2 \xrightarrow{l} d_2).
$$

We note that $F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_1 \xrightarrow{k} d_1) = r_{\mathscr{L}}^* F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_2 \xrightarrow{l} d_2)$, where $r_{\mathscr{L}}^*$ is now the pullback on sections, and see that we are looking for f_5 in

$$
\tau_! r_{\mathscr{L}}^* F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_5} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})
$$

with $\tau_!$ induced from $\tau : \pi_1 r_{\mathscr{L}} \to \pi_2$ by Lemma 1.45.

There is a base-change map

$$r_{\mathscr{L}}^* F_{c_1,*} \to F'_{c_2,*} r_{\mathscr{L}'}^*$$

with components lying the category $\mathsf{Sect}(c_2\backslash\mathscr{L}, (\pi_1 r_{\mathscr{L}})^*\mathcal{E})$. The prime over the functor $F'_{c_2,*}$ denotes that it is adjoint for the sections of the prefibration $(\pi_1 r_{\mathscr{L}})^*\mathcal{E}$ and not $\pi_2^*\mathcal{E}$. Now, apply $\tau_!$ and get

$$\tau_! r_{\mathscr{L}}^* F_{c_1,*} \to \tau_! F'_{c_2,*} r_{\mathscr{L}'}^* \to F_{c_2,*}\tau'_! r_{\mathscr{L}'}^*$$

with the second arrow existing due to the fourth statement of Lemma 1.45, with $\tau' : \pi'_1 r'_{\mathscr{L}'}\pi'_2$ be the natural transformation between the evident projections $\pi_1 : c_1\backslash\mathscr{L}' \to \mathcal{C}'$, $\pi_2 : c_2\backslash\mathscr{L}' \to \mathcal{C}'$ and $r_{\mathscr{L}'}$.

Examining what is remaining we see that to get f_5, we may as well construct f_6 in

$$F_{c_2,*}\tau'_! r_{\mathscr{L}'}^* X|_{c_1\backslash\mathscr{L}'} \xrightarrow{F_{c_2,*} f_6} F_{c_2,*} X|_{c_2\backslash\mathscr{L}'},$$

or, removing $F_{c_2,*}$,

$$\tau'_! r_{\mathscr{L}'}^* X|_{c_1\backslash\mathscr{L}'} \xrightarrow{f_6} X|_{c_2\backslash\mathscr{L}'},$$

This map, is, however, simply there by the third statement of Lemma 1.45, since X is a factual section of a semifibration. If we consider the factorisation diagram defining $r_{\mathscr{L}'}$,

$$\begin{array}{ccc}
c_1 & \xrightarrow{\ \ r\ \ } & c_2 \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
F(d_1) & \xrightarrow[F(e)]{} & F(d_2)
\end{array} \qquad\qquad (1.11)$$

then the map $f_6(b)$ corresponds to $F(e)_! X(F(d_1)) \to X(F(d_2))$. We thus get f_6 and reverse all the discussion to get f_1. $\qquad\square$

Corollary 1.49. *Let $F : \mathcal{D} \to \mathcal{C}$ be a factorisation right-closed functor such that its restriction $F_{\mathscr{L}} : \mathscr{L}' \to \mathscr{L}$ is a closed immersion of Noether categories. Then for any fibrewise-complete semifibration $\mathcal{E} \to \mathcal{C}$, there is an adjunction $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathsf{Sect}(\mathcal{D}, \mathcal{E}) : F_*$, and the right adjoint can be calculated by restricting to the left parts of the factorisation systems.*

Proof. The right adjoint for $F_{\mathscr{L}} : \mathscr{L}' \to \mathscr{L}$ exists thanks to Proposition 1.31 and is pointwise due to Proposition 1.32. $\qquad\square$

2 Reedy model structures

2.1 Model categories and localisation

Definition 2.1. A *homotopical, or relative, category*, or a *localiser*, is a pair $(\mathcal{M}, \mathcal{W})$ of a category $\mathcal{M}$ and a subcategory $\mathcal{W}$ containing all objects of $\mathcal{M}$, called the category of weak equivalences.

The definition of a model category used in this work is the following one:

Definition 2.2. A category $\mathcal{M}$ carries a *model structure*, or is a *model category*, if there are given three subcategories $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ containing all objects of $\mathcal{M}$, called respectively the subcategory of weak equivalences, cofibrations and fibrations, such that the following list of axioms is satisfied.

M1 (Property of $\mathcal{M}$) the category $\mathcal{M}$ admits small limits and colimits.

M2 The subcategory $\mathcal{W}$ satisfies 3-for-2: given two composable maps f, g, if any two elements of the set $\{f, g, gf\}$ are morphisms of $\mathcal{W}$, then so is the third.

M3 The subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are stable by retracts: given a commutative diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{i_1} & X & \xrightarrow{r_1} & A \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\
B & \xrightarrow{i_2} & Y & \xrightarrow{r_2} & B
\end{array}
$$

with $r_1 i_1 = id_A$ and $r_2 i_2 = id_B$, if g belongs to $\mathcal{W}$ (respectively to $\mathcal{C}, \mathcal{F}$), then so does f.

M4 In a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\
B & \xrightarrow{b} & Y
\end{array}
$$

with i in $\mathcal{C}$ and f in $\mathcal{F}$, whenever any of i, f is also in $\mathcal{W}$, there exists a map $p : B \to X$ with $pi = a$ and $fp = b$.

M5 Any morphism $p : X \to Y$ can be factored as $X \xrightarrow{i} Z \xrightarrow{f} Y$ with i in $\mathcal{C}$ and f in $\mathcal{F} \cap \mathcal{W}$, and as $X \xrightarrow{j} Z' \xrightarrow{g} Y$, with j in $\mathcal{C} \cap \mathcal{W}$ and g in $\mathcal{F}$.

A functor $F : \mathcal{M} \to \mathcal{N}$ of model categories is called *left-derivable* if it preserves cofibrations and trivial cofibrations, and *left Quillen* if in addition it is a left adjoint. The notions of *right-derivable* and *right Quillen* functors are defined dually.

Definition 2.3. Let $(\mathcal{M}, \mathcal{W})$ be a homotopical category. The *localisation* of $\mathcal{M}$ along $\mathcal{W}$ [8, 17], is the category which we denote $\mathcal{W}^{-1}\mathcal{M}$ or $\mathrm{Ho}\,\mathcal{M}$, together with a functor $p : \mathcal{M} \to \mathcal{W}^{-1}\mathcal{M}$ such that any functor $F : \mathcal{M} \to \mathcal{N}$ which sends $\mathcal{W}$ to isomorphisms of $\mathcal{N}$, factors through p. The factorisation is unique up to a canonical isomorphism.

Proposition 2.4 ([8, 12, 17]). *For a model category $\mathcal{M}$, the localisation $\mathrm{Ho}\,\mathcal{M}$ of $\mathcal{M}$ along $\mathcal{W}$ exists and is of the same (set-theoretical) size as $\mathcal{M}$.*

The higher-categorical localisation is revisited in Section 3.

2.2 Semifibrations over Reedy categories

Definition 2.5. Let $\mathcal{R}$ be a Reedy category. A *model semifibration* over $\mathcal{R}$ is a functor $\mathcal{E} \to \mathcal{R}$ such that it is a semifibration over $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$, each fibre $\mathcal{E}(x)$ is a model category, and

1. the transition functors along $\mathcal{R}_-$ are right derivable,

2. the transition functors along $\mathcal{R}_+$ left derivable.

We shall prove that under some admissibility conditions, the category of sections $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ carries a model structure.

Recall [12, 17] that for each object $x \in \mathcal{R}$, we have associated latching and matching categories $Lat(x)$ and $Mat(x)$. Let $\mathcal{E} \to \mathcal{R}$ be a semifibration. Then for each $x \in \mathcal{R}$, there are natural restriction functors $L_x : \mathcal{E}|_{Lat(x)} \to \mathcal{E}(x)$ and $R_x : \mathcal{E}|_{Mat(x)} \to \mathcal{E}(x)$. Indeed, by L_x, an object $X \in \mathcal{E}|_{Lat(x)}$ living over $f : y \to x$ is sent to its opcartesian image $f_! X \in \mathcal{E}(x)$, and dually for R_x.

There are choices involved in constructing L_x and R_x; both functors are unique up to a natural isomorphism.

Definition 2.6 (cf Definition 1.25). For $S \in \mathsf{Sect}(\mathcal{R}, \mathcal{E})$ and x in $\mathcal{R}$, we define the latching object of S at x to be the following colimit:

$$\mathscr{L}_x S := \varinjlim_{Lat(x)} L_x \circ S|_{Lat(x)}.$$

The matching object of S at x is defined to be the following limit:

$$\mathscr{M}_x S := \varprojlim_{Mat(x)} R_x \circ S|_{Mat(x)}.$$

The latching and matching object constructions are suitably functorial. Denote by $\mathcal{R}_{<n}$ the subcategory of objects of degree less than n. We see that $\mathscr{L}_x$ and $\mathscr{M}_x$ define functors $\mathsf{Sect}(\mathcal{R}_{<n}, \mathcal{E}) \to \mathcal{E}(x)$. Now, consider a section $S : \mathcal{R}_{<n} \to \mathcal{E}$. Then for each z of degree (up to) n, the map $\mathscr{L}_z S \to \mathscr{M}_z S$ is canonically determined. To see this, we need to supply, for each degree-raising map $g : x \to z$ and each degree-lowering map $k : z \to t$, a map $g_! S(x) \to k^* S(t)$. Since $\mathcal{R}$ is a Reedy category, we have the following square

$$
\begin{array}{ccc}
x & \xrightarrow{\ f\ } & y \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
z & \xrightarrow{\ k\ } & t
\end{array}
$$

in which the vertical maps raise the degree and the horizontal maps lower the degree. Proposition 1.40 then implies that we have a natural transformation $g_! f^* \to k^* h_!$. The sought-after map is then defined as the composition

$$g_! S(x) \to g_! f^* S(y) \to k^* h_! S(y) \to k^* S(t)$$

31

with $S(x) \to f^*S(y)$ and $h_!S(y) \to S(t)$ existing because S is a section on $\mathcal{R}_{<n}$. Combining different maps $g_!S(x) \to k^*S(t)$, we get the map from the colimit to the limit, that is, $\mathscr{L}_zS \to \mathscr{M}_zS$.

For a section $S : \mathcal{R} \to \mathcal{E}$ defined on the whole of $\mathcal{R}$, we are supplied with maps $\mathscr{L}_xS \to S(x) \to \mathscr{M}_xS$ in the fibre $\mathcal{E}(x)$ which can be seen to factor the canonical map $\mathscr{L}_xS \to \mathscr{M}_xS$.

Proposition 2.7. *Let $\mathcal{E} \to \mathcal{R}$ be a semifibration and $S : \mathcal{R}_{<n} \to \mathcal{E}$ be a section defined on objects of degree less than n. Then an extension of S to a section on objects x of degree n is equivalent to factoring the canonical maps $\mathscr{L}_xS \to \mathscr{M}_xS$ as $\mathscr{L}_xS \to S(x) \to \mathscr{M}_xS$ for each x.*

Proof. Given any non-trivial map $x \to y$ between two objects of degree n, we factor it as $x \to z \to y$, and the corresponding map $S(x) \to S(y)$ is constructed as

$$S(x) \to \mathscr{M}_xS \to S(z) \to \mathscr{L}_yS \to S(y),$$

with the middle maps well defined as $deg z < n$. $\qquad\qquad\square$

As mentioned above, the assignments $S \mapsto \mathscr{L}_xS$ and $S \mapsto \mathscr{M}_xS$ define functors from $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ to $\mathcal{E}(x)$. Thus, given a map $f : S \to T$ of two sections $S, T \in \mathsf{Sect}(\mathcal{R}, \mathcal{E})$, we get, naturally, two following squares

$$
\begin{array}{ccccc}
\mathscr{L}_xS & \longrightarrow & S(x) & \longrightarrow & \mathscr{M}_xS \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{L}_xT & \longrightarrow & T(x) & \longrightarrow & \mathscr{M}_xT
\end{array}
$$

Definition 2.8. A map of sections $f : S \to T$ is a

1. *Reedy cofibration* if the map $\mathscr{L}_xT \coprod_{\mathscr{L}_xS} S(x) \to T(x)$ is a cofibration in $\mathcal{E}(x)$ for each $x \in \mathcal{R}$.

2. *Reedy fibration* if the map $S(x) \to \mathscr{M}_xS \prod_{\mathscr{M}_xT} T(x)$ is a fibration in $\mathcal{E}(x)$ for each $x \in \mathcal{R}$.

3. *Reedy weak equivalence* if it is a fibrewise weak equivalence.

Definition 2.9. Let $\mathcal{R}$ be a Reedy category and $\mathcal{E} \to \mathcal{R}$ a model semifibration. We call $\mathcal{E} \to \mathcal{R}$ *left-admissible* if the following holds. Let $\alpha : A \to B$ be a map of sections such that $\mathscr{L}_xB \coprod_{\mathscr{L}_xA} A(x) \to B(x)$ is a (trivial) cofibration for each $x \in \mathcal{R}$. Then for any $y \in \mathcal{R}$, the map $\mathscr{L}_y(\alpha) : \mathscr{L}_yA \to \mathscr{L}_yB$ is also a (trivial) cofibration.

Dually, one can define a *right-admissible* model semifibration. A semifibration is called *admissible* if it is both left- and right-admissible.

We will show that under the left admissibility condition, the trivial Reedy cofibrations are exactly the maps $A \to B$ such that $\mathscr{L}_xB \coprod_{\mathscr{L}_xA} A(x) \to B(x)$ is a (trivial) cofibration for

each $x \in \mathcal{R}$. The left admissibility condition can be interpreted as ensuring that the functors $\mathscr{L}_x : \mathsf{Sect}(\mathcal{R}, \mathcal{E}) \to \mathcal{E}(x)$ are left derivable.

Lemma 2.10. *A model semifibration $\mathcal{E} \to \mathcal{R}$ is left-admissible if for each $x \in \mathcal{R}$, either of the following holds:*

1. *the restriction $\mathcal{E}|_{Mat(x)} \to Mat(x)$ is a fibration (and not simply a prefibration) whose transition functors preserve limits,*

2. *the category $Mat(x)$ is a disjoint union of categories possessing initial objects.*

A dual result is valid for right-admissibility.

Proof. For each y in $\mathcal{R}$, there are two possible scenarios.

1. The functor $\mathcal{E}|_{Lat(y)} \to Lat(y)$ is an opfibration (not merely a preopfibration) whose transition functors preserve colimits. The restriction to the fibre $\mathcal{E}(y)$ provides us with a functor $L : \mathsf{Sect}(Lat(y), \mathcal{E}) \to \mathsf{Fun}(Lat(y), \mathcal{E}(y))$. A section $S \in \mathsf{Sect}(Lat(y), \mathcal{E})$ is sent to
$$L(S) : (z \xrightarrow{f} y) \in Lat(y) \;\mapsto\; L(S)(f) \cong f_! S(z) \in \mathcal{E}(y).$$
The category $\mathsf{Fun}(Lat(y), \mathcal{E}(y))$ has a well-known Reedy structure [17]. Let us compute the value $Lat_f L(S)$ of the latching object functor at $f : z \to y$ (we write Lat to distinguish from $\mathscr{L}$ which we used for sections). Abusing slightly the notation, one can see that, naturally in S,
$$Lat_f L(S) \cong \varinjlim_{g:t\to z \in Lat(z)} L(S)(t \xrightarrow{g} z \xrightarrow{f} y) \cong \varinjlim_{t \to z \in Lat(z)} (fg)_! S(t)$$
$$\cong \varinjlim_{t \to z \in Lat(z)} f_! g_! S(t) \cong f_! \mathscr{L}_z(S),$$
where the last two isomorphisms are consequences of the given admissibility condition. One can use this and similar computations to verify that the image of the map $\alpha : A \to B$ in $\mathsf{Fun}(Lat(y), \mathcal{E}(y))$ is a (trivial) Reedy cofibration. Given that $\mathscr{L}_y A \cong \varinjlim_{Lat(y)} L(A)$, the necessary result follows from the classical case [17].

2. The category $Lat(y)$ is a disjoint union of categories with terminal objects. Any colimit over a such category is a coproduct of evaluations at terminal objects of the components. If we suppose that the assertion of the lemma was proven by induction for lesser degrees, the map $\mathscr{L}_y(\alpha) : \mathscr{L}_y A \to \mathscr{L}_y B$ will be represented as a coproduct $\coprod X_i \to \coprod Y_i$, where each map $X_i \to Y_i$ is a (trivial) cofibration. Thus $\mathscr{L}_y(\alpha)$ is also a (trivial) cofibration. $\square$

Theorem 2.11. *Let $\mathcal{R}$ be a Reedy category and $\mathcal{E} \to \mathcal{R}$ an admissible model semifibration. Then the category of sections $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ carries a model structure given by Reedy cofibrations, Reedy fibrations and Reedy weak equivalences of Definition 2.8.*

Remark 2.12. The admissibility condition is used in the proof of Lemma 2.18. Without admissibility, many aspects of the Reedy proof do indeed go through, but one has no control over the intersection of the classes of maps, and consequently, over factorisations and (co)fibrant replacements.

The condition (1.) and its dual of Lemma 2.10 has the property that if it is true for an object x, then it is also true for all objects y in its matching (or latching) category.

The condition (2.) and its dual of Lemma 2.10 are related to the notion of fibrant and cofibrant constants [8]. The reason that it appears in our setting is somehow dual to that of [8]; see Lemma 2.18 for details.

Lemma 2.13. *The Reedy weak equivalences are stable under retracts and satisfy the "three-for-two" axiom.*

Proof. Clear, by considering what happens in each fibre. $\qquad\square$

Lemma 2.14. *Let $f : S \to T$ be a map of sections such that f satisfies one of the properties below:*

- *For each $x \in \mathcal{R}$, the map $\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$ is a cofibration,*

- *For each $x \in \mathcal{R}$, the map $\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$ is a trivial cofibration,*

- *For each $x \in \mathcal{R}$, the map $S(x) \to \mathscr{M}_x S \prod_{\mathscr{M}_x T} T(x)$ is a fibration,*

- *For each $x \in \mathcal{R}$, the map $S(x) \to \mathscr{M}_x S \prod_{\mathscr{M}_x T} T(x)$ is a trivial fibration.*

Then any retract of f also satisfies such a property.

Proof. Let

$$
\begin{array}{ccccc}
A & \xrightarrow{\;i_1\;} & X & \xrightarrow{\;r_1\;} & A \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\
B & \xrightarrow{\;i_2\;} & Y & \xrightarrow{\;r_2\;} & B
\end{array}
$$

be a retract diagram in $Sect(\mathcal{R}, \mathcal{E})$. The assignment $A \mapsto \mathscr{L}_x A$ is functorial in A, so it preserves retracts. Then, for $x \in \mathcal{R}$, there is a diagram D_1

$$
\begin{array}{ccccc}
A(x) & \xrightarrow{\;i_1(x)\;} & X(x) & \xrightarrow{\;r_1(x)\;} & A(x) \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{L}_x A & \xrightarrow{\;\mathscr{L}_x i_1\;} & \mathscr{L}_x X & \xrightarrow{\;\mathscr{L}_x r_1\;} & \mathscr{L}_x A \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{L}_x B & \xrightarrow{\;\mathscr{L}_x i_2\;} & \mathscr{L}_x Y & \xrightarrow{\;\mathscr{L}_x r_2\;} & \mathscr{L}_x B
\end{array}
$$

which can be viewed as a retract diagram in $Fun(I, \mathcal{E}(x))$, where I is the category $0 \leftarrow 1 \rightarrow 2$. There is also a retract diagram D_2

$$B(x) \xrightarrow{i_2(x)} Y(x) \xrightarrow{r_2(x)} B(x)$$

For a category $\mathcal{D}$, let $Ret(\mathcal{D})$ be the category of retract diagrams: its objects are pairs of arrows $C \xrightarrow{i} D \xrightarrow{r} C$ with $r \circ i = id_C$. For any small category $\mathcal{J}$ the constant diagram functor $c_{\mathcal{J}}^* : \mathcal{D} \rightarrow Fun(\mathcal{J}, \mathcal{D})$ induces a functor $Ret(c_{\mathcal{J}}^*) : Ret(\mathcal{D}) \rightarrow Ret(Fun(\mathcal{J}, \mathcal{D}))$. If $\mathcal{D}$ admits small colimits, this functor has a left adjoint $Ret(\varinjlim_{\mathcal{J}}) : Ret(Fun(\mathcal{J}, \mathcal{D})) \rightarrow Ret(\mathcal{D})$.

In our case, $\mathcal{D} = \mathcal{E}(x)$ has small colimits and $\mathcal{J} = I$. In addition, $D_1 \in Ret(Fun(I, \mathcal{E}(x)))$ and $D_2 \in Ret(\mathcal{E}(x))$. The retract diagram for maps $f : A \rightarrow B$ and $g : X \rightarrow Y$ gives us a morphism $D_1 \rightarrow Ret(c_I^*)(D_2)$. Taking the adjoint to this map, we get a map of retract diagrams $Ret(\varinjlim_I)(D_1) \rightarrow D_2$, which renders the relative latching map of f at x,

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \rightarrow B(x),$$

as a retract of the relative latching map of g at x,

$$\mathscr{L}_x Y \coprod_{\mathscr{L}_x X} X(x) \rightarrow Y(x).$$

Thus if the latter map is a (trivial) cofibration, then so is the former. For the relative matching maps, the proof is dual. $\square$

Case of a direct category

We first consider the case when $\mathcal{R} = \mathcal{R}_+$ is a direct Reedy category. In this case $\mathcal{E} \rightarrow \mathcal{R}$ is an actual opfibration. Similarly, one can consider $\mathcal{R} = \mathcal{R}_-$, and work with a fibration over $\mathcal{R}$.

Proposition 2.15. *Reedy cofibrations, objectwise fibrations and objectwise weak equivalences form a model structure on* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$.

First we need to address the limit-colimit axiom.

Lemma 2.16. *For* $\mathcal{R} = \mathcal{R}_+$, *the category* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ *admits limits and colimits.*

Proof. The existence of limits is Proposition 1.18. The colimits are given by the dual of Proposition 1.27 since $\mathcal{R}_+$ is an Artin category. $\square$

Lemma 2.17. *Suppose given a diagram of sections*

$$\begin{array}{ccc} A & \longrightarrow & S \\ f \downarrow & & \downarrow p \\ B & \longrightarrow & T \end{array}$$

35

with p and objectwise fibration (respectively trivial fibration). If for each $x \in \mathcal{R}$, the map

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x) \tag{2.1}$$

is a trivial cofibration (respectively a cofibration), then the diagram admits a lift.

Proof. Proceed by induction on degree. For $deg\, x = 0$, $\mathscr{L}_x(A)$ is the initial object of $\mathcal{E}(x)$ (and the same for B), so the map (*2.1*) equals $A(x) \to B(x)$. The lift then exists simply because $\mathcal{E}(x)$ is a model category.

For $deg\, x = n$, assume that we defined the lift for all lesser degrees. For each map $\alpha : y \to x$ with $deg\, y < n$, we have the assumed lift $h_y : B(y) \to S(y)$, and the composition $B(y) \to S(y) \to S(x)$ can be factored as $\alpha_! B(y) \to S(x)$, and that in turn induces the map $\mathscr{L}_x B \to S(x)$. We then get the following diagram,

$$
\begin{array}{ccc}
\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) & \longrightarrow & S(x) \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle p} \\
B(x) & \longrightarrow & T(x)
\end{array}
$$

and we can find the necessary lift (by also remembering $A(x) \to \mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x)$). $\qquad\square$

Lemma 2.18. *Let $\alpha : A \to B$ be such that $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a (trivial) cofibration for each $x \in \mathcal{R}$. Then for any $y \in \mathcal{R}$, the maps $\mathscr{L}_y(\alpha) : \mathscr{L}_y A \to \mathscr{L}_y B$ and $\alpha_y : A(y) \to B(y)$ are (trivial) cofibrations.*

Proof. For $y \in \mathcal{R}$, note that the map $\alpha_y : A(y) \to B(y)$ equals

$$A(y) \to \mathscr{L}_y B \coprod_{\mathscr{L}_y A} A(y) \to B(y).$$

The second map is a (trivial) cofibration by condition. It thus remains to examine the map $\mathscr{L}_y(\alpha) : \mathscr{L}_y A \to \mathscr{L}_y B$. According to Definition 2.9, this map is a (trivial) cofibration, as required. $\qquad\square$

Corollary 2.19. *Let $A \to B$ be such that $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a trivial cofibration for each $x \in \mathcal{R}$. Then $A \to B$ is a Reedy cofibration and a weak equivalence.* $\qquad\square$

Proposition 2.20. *Let $A \to C$ be a map in $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$. Then it can be factored as $A \to B \to C$ where*

- *the map $A \to B$ is such that $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a cofibration (respectively a trivial cofibration) for each $x \in \mathcal{R}$,*

- *the map $B \to C$ is an objectwise trivial fibration (respectively a fibration).*

The factorisations are functorial whenever this is the case for each $\mathcal{E}(x)$.

Proof. Let us do the cofibration and trivial fibration part, the second part being dual. Factor $A(x) \to B(x)$ as $A(x) \to B(x) \to C(x)$ for each x of degree zero. Assume now that the factorisation is there for each $y \in \mathcal{R}$ of degree less than n. For x with $deg\, x = n$, we have the diagram

$$
\begin{array}{ccc}
\mathscr{L}_x A & \longrightarrow & \mathscr{L}_x B \\
\downarrow & & \downarrow \\
A(x) & \longrightarrow & C(x)
\end{array}
$$

with $\mathscr{L}_x B \to C(x)$ defined with the use of the maps $B(y) \to C(y) \to C(x)$. We thus get a map $A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B \to C(x)$, which we factor (if possible, functorially) as

$$
A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B \to B(x) \to C(x).
$$

The maps $\mathscr{L}_x B \to B(x)$ complete B to a section on $\mathcal{R}_{\leq n}$. Proceeding by induction, we get the desired factorisation. $\qquad\square$

Corollary 2.21. *A map $f : S \to T$ is a trivial Reedy cofibration iff the map*

$$
\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)
$$

is a trivial cofibration for each $x \in \mathcal{R}$.

Proof. Take a trivial Reedy cofibration $f : S \to T$ and factor it using Proposition 2.20 as $S \xrightarrow{g} U \xrightarrow{h} T$ so that $\mathscr{L}_x U \coprod_{\mathscr{L}_x S} S(x) \to U(x)$ is a trivial cofibration. We then see that f is a retract of g. $\qquad\square$

This proves the existence of the model structure on $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ for a direct category $\mathcal{R}$.

Finishing the Proof

We now turn to the case when $\mathcal{R}$ is an arbitrary Reedy category.

Lemma 2.22. *The category $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ is bicomplete.*

Proof. By Lemma 2.16 and its dual, we have that both $\mathsf{Sect}(\mathcal{R}_+, \mathcal{E})$ and $\mathsf{Sect}(\mathcal{R}_-, \mathcal{E})$ are bicomplete. The result then follows from Proposition 1.43. $\qquad\square$

Lemma 2.23. *A map $X \to Y$ is*

- *a trivial Reedy cofibration iff for each $x \in \mathcal{R}$, the map $\mathscr{L}_x Y \coprod_{\mathscr{L}_x X} X(x) \to Y(x)$ is a trivial cofibration,*

- *a trivial Reedy fibration iff for each $x \in \mathcal{R}$, the map $X(x) \to Y(x) \prod_{\mathscr{M}_x Y} \mathscr{M}_x X$ is a trivial fibration.*

Proof. For the first part, note that $X \to Y$ is a Reedy cofibration iff it is such when viewed as a morphism of sections in $\mathsf{Sect}(\mathcal{R}_+, \mathcal{E})$, since the Reedy cofibration condition is formulated objectwise in $\mathcal{R}$. It is, also, a weak equivalence iff it is such when restricted to a morphism of sections over $\mathcal{R}_+$, for the same reason. We then use Corollary 2.21 to get the result. The second part is proven in a dual manner. $\square$

Proposition 2.24. *Suppose given a diagram of sections*

$$
\begin{array}{ccc}
A & \longrightarrow & S \\
f \downarrow & & \downarrow p \\
B & \longrightarrow & T
\end{array}
$$

where $f : A \to B$ is a Reedy cofibration and $p : S \to T$ is a Reedy fibration. Then a lift exists whenever f or p is trivial.

Proof. By induction we can assume having supplied a lift for $y \in \mathcal{R}$ of degree less than n. Given an object x of degree n, we can draw the following diagram

$$
\begin{array}{ccccc}
A(x) & \longrightarrow & A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B & \longrightarrow & S(x) \\
& \searrow & \downarrow & & \swarrow \\
& & B(x) \longrightarrow T(x) \prod_{\mathscr{M}_x T} \mathscr{M}_x S \longrightarrow T(x).
\end{array}
$$

Just as in the classical case, a lift in the middle square of this diagram (which exists whenever f or p is trivial) determines the looked-for lift $B \to S$ on objects of degree n. $\square$

Proposition 2.25. *Let $A \to C$ be a map in $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$. Then it can be factored as $A \xrightarrow{i} B \xrightarrow{p} C$, with i a Reedy cofibration and p a Reedy fibration, such that either i or p is trivial. The factorisation is functorial whenever each $\mathcal{E}(x)$ admits functorial factorisations.*

Proof. Assume again that, by induction, we have constructed the factorisation $A(y) \to B(y) \to C(y)$ for objects $y \in \mathcal{R}$ of degree less than n. For x of degree n, there is the following diagram

$$
\begin{array}{ccccc}
\mathscr{L}_x A & \longrightarrow & A(x) & \longrightarrow & \mathscr{M}_x B \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{L}_x B & \longrightarrow & C(x) & \longrightarrow & \mathscr{M}_x C
\end{array}
$$

38

which exists because of the inductive assumption and provides us with the following map

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.$$

Factoring it (using the model structure of $\mathcal{E}(x)$) as

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.$$

which, together with maps $\mathscr{L}_x B \to B(x)$ and $B(x) \to \mathscr{M}_x B$, yields the desired extension of the factorisation to the objects of degree n. $\qquad\square$

We have thus proven the existence of the Reedy model structure on $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$.

Lemma 2.26. *Let $X \to Y$ be a Reedy cofibration (respectively a fibration). Then for each $x \in \mathcal{R}$, the map $X(x) \to Y(x)$ is a cofibration (respectively a fibration).*

Proof. Direct consequence of Lemma 2.18. $\qquad\square$

2.3   Cofibrant generation of Reedy structures

In this subsection we treat the situation when the semifibration $\mathcal{E} \to \mathcal{R}$ has cofibrantly generated model categories as fibres. One may ask in this case if the model category of sections is cofibrantly generated.

The Reedy category case is unorthodox in the sense that we already know the model structure, while the techniques of cofibrantly generated categories usually start with a set of generating cofibrations and a well-behaved class (large set) of weak equivalences in a presentable category to obtain a combinatorial model structure, as per Smith's theorem [18, Proposition A.2.6.8].

For the purposes of this subsection, let us introduce the following definition. As usual, write [1] for the arrow category $0 \to 1$ and $[2] = 0 \to 1 \to 2$.

Definition 2.27. A model category $\mathcal{M}$ has *accessible factorisations*, if the underlying category of $\mathcal{M}$ is presentable, and both (fibration-trivial cofibration) and (trivial fibration-cofibration) factorisations are functorial and accessible, when viewed as functors from $\mathsf{Fun}([1], \mathcal{M})$ to $\mathsf{Fun}([2], \mathcal{M})$.

Any combinatorial model category has accessible factorisations [3, Proposition 1.10]. We would like to investigate when the converse implication holds. For our purposes, the following shall suffice.

Lemma 2.28. *Let $\mathcal{M}$ be a model category with accessible factorisations, such that the subcategories of trivial fibrations and fibrations are accessible and accessibly embedded in $\mathsf{Fun}([1], \mathcal{M})$. Then $\mathcal{M}$ is cofibrantly generated, and hence is combinatorial.*

39

Proof. Choose a regular cardinal λ such that the following holds:

1. The categories $\mathcal{M}, \mathsf{Fun}([1], \mathcal{M})$ and the subcategory $\mathcal{F}ib \subset \mathsf{Fun}([1], \mathcal{M})$ consisting of fibrations (and a similar category for trivial fibrations) is λ-accessible, and the functor $\mathcal{F}ib \to \mathsf{Fun}([1], \mathcal{M})$ preserves λ-presented objects.

2. For each arrow $X \to Y$ where X and Y are λ-presentable (or λ-compact in the terminology of [18]), its factorisations

$$X \overset{\sim}{\hookrightarrow} Z \twoheadrightarrow Y \text{ and } \quad X \hookrightarrow Z' \overset{\sim}{\twoheadrightarrow} Y$$

have the property that both Z and Z' are λ-presentable. This is possible due to the accessible factorisations condition, following the same reasoning as in [7, Proposition 7.2].

The category $\mathsf{Fun}([1], \mathcal{M})$ is seen [18, Proposition 5.4.4.3] to be generated by the (essentially small) subcategory of arrows $A \to B$ where A and B are λ-presentable. We claim that the trivial cofibrations are generated by the subset $\mathcal{WC}of_\lambda$ consisting of all trivial cofibrations between λ-presentable objects.

It will suffice to check that any map $f : X \to Y$ having the right lifting property with respect to $\mathcal{WC}of_\lambda$ is a fibration. As an object of $\mathcal{M}^{[1]} = \mathsf{Fun}([1], \mathcal{M})$, the morphism f is a colimit of λ-presentable objects over a small (modulo choice), λ-filtered diagram $\mathcal{M}^{[1]}_\lambda / f$. There is also a diagram $\mathcal{F}ib_\lambda / f$ consisting of all λ-presentable objects of $\mathcal{F}ib$ together with a map in $\mathcal{M}^{[1]}$ to f. Since $\mathcal{F}ib$ is accessibly embedded we have a natural fully faithful functor

$$F : \mathcal{F}ib_\lambda / f \to \mathcal{M}^{[1]}_\lambda / f$$

We claim that F is cofinal. For this, just as in [1, Theorem 4.8], it is enough to verify that for any $g \to f$ in $\mathcal{M}^{[1]}_\lambda / f$ there exists a map $g \to F(h)$ over f for some $h \in \mathcal{F}ib_\lambda / f$.

Factoring $g : A \to B$ as a trivial cofibration followed by fibration, we get the diagram

$$
\begin{array}{ccc}
A & \overset{\sim}{\hookrightarrow} & A' \\
g \downarrow & & \downarrow F(h) \\
B & \overset{=}{\longrightarrow} & B
\end{array}
$$

the object A' is λ-presentable just like A and B, so the fibration $F(h) : A' \twoheadrightarrow B$ is λ-presentable in $\mathcal{F}ib$ as well. Moreover any map $g \to f$ can be extended in a compatible way to $F(h) \to f$, using the assumption that f has the right lifting property along trivial cofibrations between λ-presentable objects. This concludes the proof of cofinality of F, and hence also of λ-filteredness of $\mathcal{F}ib_\lambda / f$. We then use the fact that $\mathcal{F}ib$ is closed under λ-filtered colimits to conclude that $f \in \mathcal{F}ib$. The case of trivial fibrations is treated similarly. $\qquad \square$

Corollary 2.29. *Let $\mathcal{M}$ be a model category with underlying category presentable. Assume that both of the following hold:*

    i. The (cofibration-trivial fibration) factorisation is functorial and accessible,

    ii. The full subcategory $\mathcal{W}\mathcal{F}ib \subset \mathcal{M}^{[1]}$ of trivial fibrations is accessible and accessibly embedded.

Assume further that either of the following holds:

    1. The (trivial cofibration- fibration) factorisation is functorial and accessible, and the full sub-category $\mathcal{F}ib \subset \mathcal{M}^{[1]}$ of fibrations is accessible and accessibly embedded, or

    2. The full subcategory $\mathcal{W} \subset \mathcal{M}^{[1]}$ of weak equivalences is accessible and accessibly embedded.

Then $\mathcal{M}$ is combinatorial.

Proof. Combine the precedent lemma together with [18, Corollary A.2.6.9] $\square$

Proposition 2.30. *Let $\mathcal{E} \to \mathcal{R}$ be an admissible model semifibration over a Reedy category $\mathcal{R}$. Assume that each $\mathcal{E}(x)$ is combinatorial and all transition functors of the semifibration are accessible. Then the model category of sections $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ is combinatorial.*

Proof. The category $\mathcal{R}$ is a directed colimit of its subcategories $\mathcal{R}_{\leq n}$. Let us first analyse the case of finite degree.

For $\mathcal{R}_0$, the category $\mathsf{Sect}(\mathcal{R}_0, \mathcal{E})$ is the product $\prod_x \mathcal{E}(x)$ of the fibre categories over all objects of degree zero. It is, hence, combinatorial.

For $\mathcal{R}_{\leq n}$, observe that we have the following pullback diagram

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E}) & \longrightarrow & \displaystyle\prod_{deg x = n} \mathcal{E}(x)^{[2]} \\
\Big\downarrow & & \Big\downarrow \\
\mathsf{Sect}(\mathcal{R}_{<n}, \mathcal{E}) & \longrightarrow & \displaystyle\prod_{deg x = n} \mathcal{E}(x)^{[1]}.
\end{array}
\tag{2.2}
$$

The upper horizontal functor is given by $S \mapsto \mathscr{L}_x S \to S(x) \to \mathscr{M}_x S$ for all objects x of degree n. The bottom horizontal functor is similarly given by $S \mapsto \mathscr{L}_x S \to \mathscr{M}_x S$. The vertical functors are the evident restrictions.

Both categories $\prod_x \mathcal{E}(x)^{[2]}$ and $\prod_x \mathcal{E}(x)^{[1]}$ are presentable. By induction, we can assume that $\mathsf{Sect}(\mathcal{R}_{<n}, \mathcal{E})$ are combinatorial and that both $\mathscr{L}_x$ and $\mathscr{M}_x$ are accessible functors (the initialization is given by $\emptyset = \mathcal{R}_{-1} \subset \mathcal{R}_0$). This implies that the horizontal functors in (2.2) are accessible. We thus see that $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$ is accessible, and hence presentable. We also infer, by induction, that sufficiently large filtered colimits are computed fibrewise in $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$.

We shall not treat $\mathcal{E}(x)^{[1]}$ and $\mathcal{E}(x)^{[2]}$ as model category. However, consider the full subcategory $\mathcal{F}ib(x)^{[2]} \subset \mathsf{Fun}([1], \mathcal{E}(x)^{[2]})$ given by all diagrams

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z
\end{array}
$$

such that both $C \to Z$ and $B \to Y \times_Z C$ are fibrations. The category $\mathcal{F}ib(x)^{[2]}$ is the pullback, along the inclusion $[1] \subset [2]$ skipping 0, of the subcategory of fibrations for the inverse Reedy model structure on each $\mathcal{E}(x)^{[1]}$. The latter is well known to be combinatorial (it follows for example from Proposition 2.32 below). Using [18, Corollary A.2.6.9] and the closure of accessible categories under limits, we obtain that $\mathcal{F}ib(x)^{[2]}$ is accessible and accessibly embedded. In a similar vein, we can define an accessible and accessibly embedded subcategory $\mathcal{F}ib(x)^{[1]} \subset \mathsf{Fun}([1], \mathcal{E}(x)^{[1]})$ to be given by all diagrams

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

such that $C \to Z$ is a fibration.

We then conclude, using the diagram (*2.2*), that the category of fibrations of $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$ (and, similarly, of trivial fibrations) is accessible and accessibly embedded, since we can express it as a limit of $\prod_x \mathcal{F}ib(x)^{[2]}$, $\prod_x \mathcal{F}ib(x)^{[1]}$ and the fibrations of $\mathsf{Sect}(\mathcal{R}_{<n}, \mathcal{E})$ along accessible functors. This means that the fibrations (and, similarly, trivial fibrations) form an accessible (and accessibly embedded) subcategory of $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})^{[1]}$. We also see that the factorisations in $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$ are functorial and accessible: they are obtained through the inductive procedure involving lesser degree latching and matching objects, fibred products and pushouts, and factorisations in the fibres.

Now, given an object y of the successive degree, consider the matching object functor $\mathscr{M}_y : \mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E}) \to \mathcal{E}(y)$. It is obtained as the composition of the restriction to the fibre

$$
\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E}) \to \mathsf{Sect}(Mat(y), \mathcal{E}) \to \mathsf{Fun}(Mat(y), \mathcal{E}(y))
$$

with the $Mat(y)$-limit functor. Bearing in mind that we proved that (sufficiently large) filtered colimits are computed fibrewise in $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$, all of these functors are accessible. We thus obtain that $\mathscr{M}_y$ is accessible. One can prove similar for $\mathscr{L}_y$.

All this shows that for each degree n, the category $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$ is a combinatorial model category. The restriction functors $\mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{R}_{\leq n-1}, \mathcal{E})$ commute with colimits. Using the argumentation as before, but with pullbacks of categories replaced by directed colimits, we obtain that $\mathsf{Sect}(\mathcal{R}, \mathcal{E}) = \varprojlim \mathsf{Sect}(\mathcal{R}_{\leq n}, \mathcal{E})$ satisfies the conditions of Lemma 2.28. It is hence combinatorial. $\qquad\square$

We conclude this subsection by considering the case when $\mathcal{E} \to \mathcal{R}$ is a bifibration, that is, both a Grothendieck fibration and an opfibration. The assumptions of the model semifibration (which in this case is automatically admissible due to (1.) of Lemma 2.10) imply that for any $f : x \to y$, the induced adjunction

$$f_! : \mathcal{E}(x) \rightleftarrows \mathcal{E}(y) : f^*$$
<div align="right">(2.3)</div>

is a Quillen pair. In the terminology of [16], we are dealing with a *Quillen presheaf*.

Let $x \in \mathcal{R}$ and $\mathcal{C}$ be a subcategory of the comma category $x \backslash \mathcal{R}$. It comes equipped with a natural functor $p_{\mathcal{C}} : \mathcal{C} \to \mathcal{R}$. This functor induces the following adjunction:

$$p_{\mathcal{C},!} : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathsf{Sect}(\mathcal{R}, \mathcal{E}) : p_{\mathcal{C}}^*.$$

The left adjoint exists in the case of a fibrewise cocomplete bifibration and sends S to the section determined by

$$p_{\mathcal{C},!}S(y) = \varinjlim_{\mathcal{C}/y} Res_y S|_{\mathcal{C}/y}$$

with $Res_y : \mathcal{E}|_{\mathcal{C}/y} \to \mathcal{E}(y)$ being the usual restriction functor. On the other hand, there is also an adjunction

$$triv_{\mathcal{C}} : \mathcal{E}(x) \rightleftarrows \mathsf{Sect}(\mathcal{C}, \mathcal{E}) : \mathscr{M}_{\mathcal{C}}.$$

The functor $triv_{\mathcal{C}}$ sends $X \in \mathcal{E}(x)$ to the section

$$(f : x \to y) \in \mathcal{C} \quad \mapsto \quad f_! X \in \mathcal{E}(y).$$

The functor $\mathscr{M}_{\mathcal{C}}$ is the composition $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{E}(x)) \to \mathcal{E}(x)$ where the first functor is induced by restrictions along cartesian morphisms, and the second is the limit functor. We thus have the composed adjunction

$$p_{\mathcal{C},!}triv_{\mathcal{C}} : \mathcal{E}(x) \rightleftarrows \mathsf{Sect}(\mathcal{R}, \mathcal{E}) : \mathscr{M}_{\mathcal{C}}p_{\mathcal{C}}^*.$$

Lemma 2.31. *For a Quillen presheaf $\mathcal{E} \to \mathcal{R}$, we have the following:*

1. *For each object $X \in \mathcal{E}(x)$, the section*

$$i(X) = p_{x \backslash \mathcal{R},!}triv_{x \backslash \mathcal{R}}X$$

   satisfies the property

$$\mathsf{Sect}(\mathcal{R}, \mathcal{E})(i(X), S) \cong \mathcal{E}(x)(X, S(x)).$$

2. *For each object $X \in \mathcal{E}(x)$, the section*

$$m(X) = p_{Mat(x),!}triv_{Mat(x)}X$$

   satisfies the property

$$\mathsf{Sect}(\mathcal{R}, \mathcal{E})(m(X), S) \cong \mathcal{E}(x)(X, \mathscr{M}_x S).$$

43

3. For each object $X \in \mathcal{E}(x)$, we have a natural map $m(X) \to i(X)$.

Proof. Immediate. $\qquad\square$

Proposition 2.32. *Let $\mathcal{E} \to \mathcal{R}$ be a Quillen presheaf. Assume that each model category $\mathcal{E}(x)$ is cofibrantly generated, with generating cofibrations denoted I_x and generating trivial cofibrations denoted J_x. Write*

$$I = \{m(B) \coprod_{m(A)} i(A) \to i(B) | A \to B \in I_x, x \in \mathcal{R}\},$$

$$J = \{m(B) \coprod_{m(A)} i(A) \to i(B) | A \to B \in J_x, x \in \mathcal{R}\}.$$

Then I and J are sets of generating cofibrations and trivial cofibrations for the model structure of Theorem 2.11 on $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$.

Proof. The usual adjunction observation, amounting to the fact that a diagram

$$
\begin{array}{ccc}
m(B) \coprod_{m(A)} i(A) & \longrightarrow & S \\
\downarrow & & \downarrow \\
i(B) & \longrightarrow & T
\end{array}
$$

is the same data as the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & S(x) \\
\downarrow & & \downarrow \\
B & \longrightarrow & T(x) \prod_{\mathscr{M}_x T} \mathscr{M}_x S
\end{array}
$$

$\qquad\square$

3   Comparing with higher sections

Definition 3.1. A *left model Reedy fibration* is a functor $p : \mathcal{E} \to \mathcal{R}$ to a Reedy category that is a Grothendieck opfibration, a Grothendieck fibration over $\mathcal{R}_-$, such that the associated semifibration (cf Lemma 1.41) is admissible in the sense of Definition 2.9.

Recall the notion of an $(\infty, 1)$-category, which can be modeled using quasicategories (called "infinity-categories" in [18]). Denote by $\mathcal{W} \subset \mathcal{E}$ the collection of those maps $\alpha : X \to Y$ such that $p\alpha$ is an isomorphism and the induced map $(p\alpha)_! X \to Y$ is a fibrewise weak equivalence. Using the localisation in quasicategories [11], localising along $\mathcal{W} \subset \mathcal{E}$ yields an infinity-functor

$Lp : L\mathcal{E} \to \mathcal{R}$ (we make no distinction between an ordinary category and its nerve in **SSet**), which can be chosen to be a categorical fibration. The associated quasicategory of sections $\mathsf{Sect}(\mathcal{R}, L\mathcal{E})$ is given by the sub-quasicategory of $\mathsf{Fun}(\mathcal{R}, L\mathcal{E})$ spanned by those S such that $Lp \circ S$ is an identity; more precisely $\mathsf{Sect}(\mathcal{R}, L\mathcal{E})$ is the fibre of $Lp_* : \mathsf{Fun}(\mathcal{R}, L\mathcal{E}) \to \mathsf{Fun}(\mathcal{R}, \mathcal{R})$ over the identity functor.

The goal of this subsection is to prove the strictification result, asserting that the infinity-localisation $L\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ of the category of sections coincides with $\mathsf{Sect}(\mathcal{R}, L\mathcal{E})$. After proving this, we shall revisit the case of Quillen presheaves — those $\mathcal{E} \to \mathcal{C}$ that are bifibrations in model categories and Quillen adjunctions — and show that the strictification holds over an arbitrary base $\mathcal{C}$.

Remark 3.2. The terminology employed in [18] is different from the one which we have used up to this point. Lurie uses the term "coCartesian fibration" [18, Definition 2.4.2.1] for what we would have called "opfibration of quasicategories", and "Cartesian fibration" for fibrations. We have chosen to stick to the terminology of [18] when dealing with higher-categorical fibrations, and we use our terminology for the 1-categorical objects.

We also specify that for us, the term infinity-localisation means a functor $F : \mathcal{X} \to \mathcal{Y}$ of quasicategories, such that for any quasicategory $\mathcal{Z}$, the induced infinity-functor $F^* : \mathsf{Fun}(\mathcal{Y}, \mathcal{Z}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{Z})$ is full and faithful, and its essential image consists of functors that send the F-equivalences of $\mathcal{X}$ to equivalences of $\mathcal{Z}$. This is not the same meaning as the localisation used in the setting of presentable infinity-categories [18, Chapter 5].

For any Reedy category, one can introduce the notion of a (transfinite) good filtration $\{\mathcal{R}_\beta\}$ [18, Notation A.2.9.11] for a Reedy category $\mathcal{R}$ (a Reedy category in our sense is a Reedy category in the sense of Lurie). The advantage of a good filtration is that, for an ordinal β, one obtains $\mathcal{R}_\beta$ by adjoining a single object to $\mathcal{R}_{<\beta} = \cup_{\gamma<\beta}\mathcal{R}_\gamma$.

The key observation [18, Proposition A.2.9.14] for Reedy categories in higher-categorical setting asserts that the diagram

$$
\begin{array}{ccc}
(\mathcal{R}_{<\beta}/x) \star (x\backslash\mathcal{R}_{<\beta}) & \longrightarrow & \mathcal{R}_{<\beta} \\
\downarrow & & \downarrow \\
(\mathcal{R}_{<\beta}/x) \star \{x\} \star (x\backslash\mathcal{R}_{<\beta}) & \longrightarrow & \mathcal{R}_\beta
\end{array}
$$

is a homotopy pushout square of quasicategories. We modify this statement slightly.

Proposition 3.3. *Let $\mathcal{R}$ be a Reedy category, and $\mathcal{R}_\beta$ be a step of a good filtration obtained from $\mathcal{R}_{<\beta}$ by adding an object x. Then the following square*

$$
\begin{array}{ccc}
Lat(x) \star Mat(x) & \longrightarrow & \mathcal{R}_{<\beta} \\
\downarrow & & \downarrow \\
Lat(x) \star \{x\} \star Mat(x) & \longrightarrow & \mathcal{R}_\beta
\end{array}
$$

is a homotopy pushout for the Joyal model structure.

Proof. Observe that the inclusion $Lat(x) \subset \mathcal{R}_{<\beta}/x$ is a right adjoint, with the left adjoint obtained by using the factorisation system. This inclusion is hence homotopy cofinal, as follows immediately from [18, Theorem 4.1.3.1]. Propositions 4.1.2.5 and 4.1.2.1 of [18] imply that $Lat(x) \to \subset \mathcal{R}/x$ is right anodyne. Taking the pushout-join with $\emptyset \subset \{x\}$ and applying [18, Lemma 2.1.2.3] yields that the diagram

$$
\begin{array}{ccc}
Lat(x) & \longrightarrow & (\mathcal{R}_{<\beta}/x) \\
\downarrow & & \downarrow \\
Lat(x) \star \{x\} & \longrightarrow & (\mathcal{R}_{<\beta}/x) \star \{x\}
\end{array}
$$

is a homotopy pushout for the Joyal model structure. Using the fact that joins preserve connected homotopy colimits (see e.g. [27, Lemma 4.14] for a relative version) and a dual argument for the inclusion $Mat(x) \subset x\backslash\mathcal{R}_{<\beta}$, we conclude that we have a series of homotopy pushout squares

$$
\begin{array}{ccccc}
Lat(x) \star Mat(x) & \longrightarrow & (\mathcal{R}_{<\beta}/x) \star Mat(x) & \longrightarrow & (\mathcal{R}_{<\beta}/x) \star (x\backslash\mathcal{R}_{<\beta}) \\
\downarrow & & \downarrow & & \downarrow \\
Lat(x) \star \{x\} \star Mat(x) & \longrightarrow & (\mathcal{R}_{<\beta}/x) \star \{x\} \star Mat(x) & \longrightarrow & (\mathcal{R}_{<\beta}/x) \star \{x\} \star (x\backslash\mathcal{R}_{<\beta})
\end{array}
$$

which together with [18, Proposition A.2.9.14] and pasting for homotopy pullbacks implies the result. $\qquad\square$

It will be easiest to state the results in the infinity-category of all infinity-categories $\mathbf{Cat}_\infty$, with usual size issue remarks. The reason for this is that many of the proofs in this section will manipulate with diagrams which are canonically presented in $\mathbf{SSet}$, but in which some categorical equivalences are pointing in the wrong direction. Such zig-zags of diagrams will induce well-defined diagrams in $\mathbf{Cat}_\infty$:

Lemma 3.4. *Assume given a zig-zag of diagrams $I \times [1] \to \mathbf{SSet}$, valued in quasicategories and depicted schematically as*

$$
\begin{array}{ccccc}
X_\bullet(0) & \xleftarrow{\sim} \dots \longrightarrow & Y_\bullet(0) \\
\downarrow & \dots & \downarrow \\
X_\bullet(1) & \xleftarrow{\sim} \dots \longrightarrow & Y_\bullet(1)
\end{array}
$$

with the bottom index corresponding to the I-direction. Assume that each left-pointing arrow is a categorical equivalence. Then there exists an induced commutative diagram in $\mathbf{Cat}_\infty$,

$$
\begin{array}{ccc}
X_\bullet(0) & \longrightarrow & Y_\bullet(0) \\
\downarrow & & \downarrow \\
X_\bullet(1) & \longrightarrow & Y_\bullet(1),
\end{array}
$$

with each horizontal arrow induced by choosing inverses of left-pointing arrows and compositions.

Proof. Direct consequence of [6, Corollary 3.5.12] applied to $\mathbf{Cat}_\infty$ itself. $\square$

We will often identify an $\mathbf{SSet}$-diagram with its image in $\mathbf{Cat}_\infty$ if the details are clear from the context.

The following proposition is the expression of the Reedy induction as applied to a suitably bicomplete infinity-category.

Proposition 3.5. *(Cf [18, Remark A.2.9.16]) let $\mathcal{R}$ be a Reedy category and $\mathcal{Y}$ be a quasicategory. Assume that $\mathcal{Y}$ admits $Lat(y)$-colimits and $Mat(y)$-limits for each $y \in \mathcal{R}$. Then, in the usual notation for a good filtration, there is a diagram in $\mathbf{Cat}_\infty$*

$$
\begin{array}{ccc}
\mathsf{Fun}(\mathcal{R}_\beta, \mathcal{Y}) & \longrightarrow & \mathsf{Fun}(\mathcal{R}_{<\beta}, \mathcal{Y}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}(Lat(x) \star \{x\} \star Mat(x), \mathcal{Y}) \longrightarrow \mathsf{Fun}(Lat(x) \star Mat(x), \mathcal{Y}) & & (3.1) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([2], \mathcal{Y}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{Y})
\end{array}
$$

with all squares pullbacks. The upper vertical functors are restrictions, the bottom left vertical functor sends S to $\varinjlim_{Lat(x)} S \to S(x) \to \varprojlim_{Mat(x)} S$, and the bottom right vertical functor sends S' to $\varinjlim_{Lat(x)} S' \to \varprojlim_{Mat(x)} S'$.

Proof. The nontrivial square of the diagram (3.1) is the bottom one. Let us correctly construct the functors appearing in it. Consider the inclusion $Lat(x) \star \{x\} \star Mat(x) \subset Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft$. The induced restriction functor

$$
\mathsf{Fun}(Lat(x) \star \{x\} \star Mat(x), \mathcal{Y}) \longleftarrow \mathsf{Fun}(Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft, \mathcal{Y})
$$

admits a section, given by putting a colimit in the cone vertex of $Lat(x)^\triangleright$ and a limit in the cone vertex of $Mat(x)^\triangleleft$. Formally, it is a composition of a left then right Kan extension along the full inclusions

$$
Lat(x) \star \{x\} \star Mat(x) \subset Lat(x)^\triangleright \star \{x\} \star Mat(x) \subset Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft.
$$

The results of [18, 4.3.2] then imply that the induced functor

$$
\mathsf{Fun}(Lat(x) \star \{x\} \star Mat(x), \mathcal{Y}) \longleftarrow \mathsf{Fun}'(Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft, \mathcal{Y})
$$

is an equivalence of infinity-categories, where $\mathsf{Fun}'(Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft, \mathcal{Y})$ is the full subcategory consisting of functors $F : Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft \to \mathcal{Y}$ which carry the cone vertex $l \in Lat(x)^\triangleright$ to the colimit of F over $Lat(x)$, and the cone vertex $m \in Mat(x)^\triangleleft$

to the limit of F over $Mat(x)$. One can apply exactly the same analysis to the inclusion $Lat(x) \star Mat(x) \subset Lat(x)^{\triangleright} \star Mat(x)^{\triangleleft}$.

The following will suffice. For any two categories A, B, we need to show that the diagram

$$\begin{array}{ccc} \mathsf{Fun}(A^{\triangleright} \star \{x\} \star B^{\triangleleft}, \mathcal{Y}) & \longrightarrow & \mathsf{Fun}([2], \mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathsf{Fun}(A^{\triangleright} \star B^{\triangleleft}, \mathcal{Y}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{Y}) \end{array}$$

is a pullback square in $\mathbf{Cat}_{\infty}$. If we denote by $a \in A^{\triangleright}$ and $b \in B^{\triangleleft}$ the cone vertices, then the horizontal arrows are induced by $[2] \cong \{a\} \star \{x\} \star \{b\} \subset A^{\triangleright} \star \{x\} \star B^{\triangleleft}$ and $[1] \cong \{a\} \star \{b\} \subset A^{\triangleright} \star B^{\triangleleft}$ respectively, with vertical arrows then also being the obvious restrictions.

In turn, it will suffice to prove that the diagram

$$\begin{array}{ccc} \{a\} \star \{b\} & \longrightarrow & A^{\triangleright} \star B^{\triangleleft} \\ \downarrow & & \downarrow \\ \{a\} \star \{x\} \star \{b\} & \longrightarrow & A^{\triangleright} \star \{x\} \star B^{\triangleleft} \end{array} \qquad (3.2)$$

is a pushout in $\mathbf{Cat}_{\infty}$. To see that the latter is true, assume first that $A = \emptyset$. Then the diagram (3.2) is a join with $\{a\} \star -$ of

$$\begin{array}{ccc} \{b\} & \longrightarrow & B^{\triangleleft} \\ \downarrow & & \downarrow \\ \{x\} \star \{b\} & \longrightarrow & \{x\} \star B^{\triangleleft} \end{array} \qquad (3.3)$$

and this diagram is the pushout-join of $\emptyset \subset \{x\}$ with $\{b\} \subset B^{\triangleleft}$. The latter is well-known to be left anodyne: it is a left adjoint, and one can thus use the same reasoning as in Proposition 3.3. Hence the diagram (3.3) is infinity-coCartesian in $\mathbf{Cat}_{\infty}$ by [18, Lemma 2.1.2.3]. We now present the diagram (3.2) as

$$\begin{array}{ccccc} \{a\} \star \{b\} & \longrightarrow & \{a\} \star B^{\triangleleft} & \longrightarrow & A^{\triangleright} \star B^{\triangleleft} \\ \downarrow & & \downarrow & & \downarrow \\ \{a\} \star \{x\} \star \{b\} & \longrightarrow & \{a\} \star \{x\} \star B^{\triangleleft} & \longrightarrow & A^{\triangleright} \star \{x\} \star B^{\triangleleft}. \end{array}$$

By using the fact that joins preserve connected homotopy colimits [27, Lemma 4.14] and the argument for the diagram (3.3) and its dual for $B = \emptyset$, we conclude that (3.2) is a pushout in $\mathbf{Cat}_{\infty}$.

To recapitulate, we have the following diagram in $\mathbf{Cat}_{\infty}$ (canonically induced from a dia-

gram in **SSet**)

$$\begin{array}{ccc}
\mathsf{Fun}(Lat(x) \star \{x\} \star Mat(x), \mathcal{Y}) & \longrightarrow & \mathsf{Fun}(Lat(x) \star Mat(x), \mathcal{Y}) \\
{\sim}\uparrow & & {\sim}\uparrow \\
\mathsf{Fun}'(Lat(x)^{\triangleright} \star \{x\} \star Mat(x)^{\triangleleft}, \mathcal{Y}) & \longrightarrow & \mathsf{Fun}'(Lat(x)^{\triangleright} \star Mat(x)^{\triangleleft}, \mathcal{Y}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}(Lat(x)^{\triangleright} \star \{x\} \star Mat(x)^{\triangleleft}, \mathcal{Y}) & \longrightarrow & \mathsf{Fun}(Lat(x)^{\triangleright} \star Mat(x)^{\triangleleft}, \mathcal{Y}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([2], \mathcal{Y}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{Y})
\end{array} \qquad (3.4)$$

and inverting in $\mathbf{Cat}_\infty$ the upper equivalences, we get the bottom square of (*3.1*). $\qquad\square$

We would like to also address the functoriality of the diagram (*3.1*) of Proposition 3.5. A typical functor which we would be interested in is the projection to the infinity-localisation $p : \mathcal{M} \to L\mathcal{M}$ of a model category $\mathcal{M}$. Given a diagram $X : Lat(x) \to \mathcal{M}$, the image of $\varinjlim X$ in $L\mathcal{M}$ ceases in general to be the colimit of $p \circ X$. It remains such, however, if X is Reedy-cofibrant [2, Proposition 2.5.6]. This motivates the following definition.

Definition 3.6. Let $\mathcal{R}$ be a Reedy category and $F : \mathcal{X} \to \mathcal{Y}$ be an infinity-functor between quasicategories both admitting $Lat(x)$-colimits and $Mat(x)$-limits for x. Let $\mathcal{R}_\beta$ be the step of a good filtration that adds x. A diagram $X : \mathcal{R}_{<\beta} \to \mathcal{X}$ is called (F, x)-*compatible* if the following holds:

i. The colimit cone $Lat(x)^{\triangleright} \to \mathcal{X}$ obtained by restricting X to $Lat(x)$ and then taking the colimit, remains a colimit cone after postcomposing with F,

ii. The limit cone $Mat(x)^{\triangleleft} \to \mathcal{X}$ obtained by restricting X to $Mat(x)$ and then taking the limit, remains a limit cone after postcomposing with F.

A general diagram $\mathcal{R} \to \mathcal{X}$ is (F, x)-compatible iff its restriction to $\mathcal{R}_{<\beta}$ is such. Denote by $\mathsf{Fun}_{(F,x)}(\mathcal{R}_{<\beta}, \mathcal{X})$, $\mathsf{Fun}_{(F,x)}(\mathcal{R}_\beta, \mathcal{X})$ and $\mathsf{Fun}_{(F,x)}(\mathcal{R}, \mathcal{X})$ the full subcategories spanned by (F, x)-compatible functors.

Proposition 3.7. *Let $\mathcal{R}$ be a Reedy category and and $F : \mathcal{X} \to \mathcal{Y}$ be an infinity-functor between quasicategories both admitting $Lat(y)$-colimits and $Mat(y)$-limits for each y in $\mathcal{R}$. Then, if $\mathcal{R}_\beta$ is obtained from $\mathcal{R}_{<\beta}$ by adding x, the diagram (3.1) of Proposition 3.5 induces the diagram $\mathcal{D}ia(\mathcal{R}_\beta, F)$:*

$$\begin{array}{ccc}
\mathsf{Fun}_{(F,x)}(\mathcal{R}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([2], \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}_{(F,x)}(\mathcal{R}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{X});
\end{array} \qquad (3.5)$$

the upper horizontal functor sends S to $\varinjlim_{Lat(x)} S \to S(x) \to \varprojlim_{Mat(x)} S$, and the bottom horizontal functor sends S' to $\varinjlim_{Lat(x)} S' \to \varprojlim_{Mat(x)} S'$.

Moreover, we have an induced natural transformation of diagrams

$$F_* : \mathcal{D}ia(\mathcal{R}_\beta, F) \longrightarrow \mathcal{D}ia(\mathcal{R}_\beta, Id_{\mathcal{Y}})$$

with $\mathcal{D}ia(\mathcal{R}_\beta, Id_{\mathcal{Y}})$ denoting the diagram (3.1) for $\mathcal{Y}$.

Proof. The definition of compatibility implies that the following square is pullback in **SSet**:

$$
\begin{array}{ccc}
\mathsf{Fun}_{(F,x)}(\mathcal{R}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Fun}(\mathcal{R}_\beta, \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}_{(F,x)}(\mathcal{R}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Fun}(\mathcal{R}_{<\beta}, \mathcal{X});
\end{array}
$$

the right functor is moreover a categorical fibration, since $\mathcal{R}_{<\beta} \subset \mathcal{R}_\beta$ is an inclusion. This diagram is hence a homotopy pullback, so the existence of (3.5) is clear. To see the functoriality, write the diagram (3.1) as a zig-zag, using the diagram (3.4). A close inspection shows that $F : \mathcal{X} \to \mathcal{Y}$ induces a map between the zig-zags representing (3.5) and (3.1): the nontrivial moment is in observing that the restriction of the functor

$$\mathsf{Fun}'(Lat(x)^{\triangleright} \star \{x\} \star Mat(x)^{\triangleleft}, \mathcal{X}) \to \mathsf{Fun}(Lat(x)^{\triangleright} \star \{x\} \star Mat(x)^{\triangleleft}, \mathcal{X}')$$

to the diagrams $Lat(x)^{\triangleright} \star \{x\} \star Mat(x)^{\triangleleft} \to \mathcal{X}$ which come from (F, x)-compatible diagrams, factors through $\mathsf{Fun}'(Lat(x)^{\triangleright} \star \{x\} \star Mat(x)^{\triangleleft}, \mathcal{X}')$. $\qquad\square$

3.1 Induction for higher-categorical sections

We need to introduce the notion of the higher-categorical restriction to the fibre. For any object $x \in \mathcal{R}$, the natural inclusion functor $Lat(x) \subset \mathcal{R}$ extends to a functor

$$C_x : Lat(x) \times [1] \to \mathcal{R}$$

which sends $(y \to x, 0)$ to y and $(y \to x, 1)$ to x. There is a similar extension for $Mat(x) \subset \mathcal{R}$.

Definition 3.8. Let $\mathcal{X} \to \mathcal{R}$ be a coCartesian fibration and $x \in \mathcal{R}$. A x-*left restriction* of $S \in \mathsf{Sect}(\mathcal{R}, \mathcal{X})$ is the infinity-functor $L_x S : Lat(x) \to \mathcal{X}(x)$ defined as follows. In the diagram in the diagram

$$
\begin{array}{ccc}
Lat(x) & \xrightarrow{\ S|_{Lat(x)}\ } & \mathcal{X} \\
{\scriptstyle id \times \{0\}} \uparrow & & \downarrow \\
Lat(x) \times [1] & \xrightarrow[\ C_x\ ]{} & \mathcal{R}
\end{array}
$$

50

choose a lifting $C_x S : Lat(x) \times [1] \to \mathfrak{X}$ which sends each map of the form $(y \to x, 0) \to (y \to x, 1)$ to a coCartesian morphism of $\mathfrak{X}$. Then $L_x S$ is the restriction of $C_x S$ to $Lat(x) \times \{1\}$.

The definition of a *x-right restriction* $R_x S : Mat(x) \to \mathfrak{X}(x)$ for a Cartesian fibration $\mathfrak{X} \to \mathcal{R}$ is given dually.

Arbitrary x-left and x-right restrictions always exist, and are defined up to an equivalence:

Lemma 3.9. *For a coCartesian fibration $\mathfrak{X} \to \mathcal{R}$, the restriction functor $\mathsf{Sect}(Lat(x) \times [1], \mathfrak{X}) \to \mathsf{Sect}(Lat(x), \mathfrak{X})$ induces an equivalence between $\mathsf{Sect}(Lat(x), \mathfrak{X})$ and a full subcategory*

$$\mathsf{Sect}_{[1]-cart}(Lat(x) \times [1], \mathfrak{X}) \subset \mathsf{Sect}(Lat(x) \times [1], \mathfrak{X})$$

consisting of all sections that send maps of the form $(y \to x, 0) \to (y \to x, 1)$ to coCartesian morphisms of $\mathfrak{X}$. There is a dual statement for Cartesian fibrations.

Proof. Recall the notion of right marked anodyne map (called marked anodyne in [18, Definition 3.1.1.1]), and the dual notion of left marked anodyne map. The inclusion $Lat(x)^\flat \hookrightarrow Lat(x)^\flat \times [1]^\sharp$ is left marked anodyne, which follows from the dual of [18, Proposition 3.1.2.3]. Observe that the functor $\mathsf{Sect}_{[1]-cart}(Lat(x) \times [1], \mathfrak{X}) \to \mathsf{Sect}(Lat(x), \mathfrak{X})$ is precisely given by the equivalence [18, Remark 3.1.3.4]

$$\mathrm{Map}_{\mathcal{R}}^\flat(Lat(x)^\flat \times [1]^\sharp, \mathfrak{X}^\natural) \overset{\sim}{\to} \mathrm{Map}_{\mathcal{R}}^\flat(Lat(x)^\flat, \mathfrak{X}^\natural),$$

where $\mathfrak{X}^\natural$ means that we mark the coCartesian arrows in $\mathfrak{X}$. Thus liftings $C_x S$ in the diagram

$$
\begin{array}{ccc}
Lat(x)^\flat & \xrightarrow{\;\; S|_{Lat(x)} \;\;} & \mathfrak{X}^\natural \\
{\scriptstyle id \times \{0\}} \downarrow \;\; & \overset{C_x S}{\nearrow} & \;\; \downarrow \\
Lat(x)^\flat \times [1]^\sharp & \xrightarrow[\;\; C_x \;\;]{} & \mathcal{R}^\sharp
\end{array}
$$

always exist and have the required property, and the space of such liftings is contractible. $\qquad\square$

Lemma 3.10. *Let $F : \mathfrak{X} \to \mathcal{Y}$ be a coCartesian morphism between coCartesian fibrations over $\mathcal{R}$. Then F preserves x-left restrictions for each $x \in \mathcal{R}$.*

Proof. The lemma follows from the existence and commutativity of the following diagram:

$$
\begin{array}{ccccc}
\mathsf{Sect}(Lat(x), \mathfrak{X}) & \overset{\sim}{\leftarrow} & \mathsf{Sect}_{[1]-cart}(Lat(x) \times [1], \mathfrak{X}) & \xrightarrow{\;ev_1\;} & \mathsf{Fun}(Lat(x), \mathfrak{X}(x)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Sect}(Lat(x), \mathcal{Y}) & \overset{\sim}{\leftarrow} & \mathsf{Sect}_{[1]-cart}(Lat(x) \times [1], \mathcal{Y}) & \xrightarrow{\;ev_1\;} & \mathsf{Fun}(Lat(x), \mathcal{Y}(x)).
\end{array}
$$

$\qquad\square$

It will be illustrative to first start with the case of a direct Reedy category $\mathcal{D}$ and a coCartesian fibration of quasicategories $\mathfrak{X} \to \mathcal{D}$.

Lemma 3.11. *Let $S \in \mathbf{SSet}$ and $i : A \to B$ be a trivial cofibration (for the Joyal model structure) in $\mathbf{SSet}/S$. Then for any subset $\mathcal{E} \in A(1)$ containing all degenerate edges, the induced map of marked simplicial sets, $(A, \mathcal{E}) \to (B, i\mathcal{E})$, is a trivial cofibration in both Cartesian and coCartesian model structures on $\mathbf{SSet}_+/S^\sharp$.*

Proof. From [18, Proposition 3.1.5.3] we know that the functor $(-)^\flat : \mathbf{SSet}/S \to \mathbf{SSet}/S^\sharp$ is left Quillen, for either Cartesian or coCartesian model structure (one can check that the proof is self-dual). We then have the following diagram in $\mathbf{SSet}/S^\sharp$

$$
\begin{array}{ccccc}
\mathcal{E} \times [1]^\flat & \longrightarrow & A^\flat & \longrightarrow & B^\flat \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E} \times [1]^\sharp & \longrightarrow & (A, \mathcal{E}) & \longrightarrow & (B, i\mathcal{E})
\end{array}
$$

where for the leftmost arrow, we equip $\mathcal{E} \times [1]$ with the map to S induced by adjunction from $\mathcal{E} \to S_1$. The leftmost and the outer squares of this diagram are pushouts, hence the same is true for the rightmost square. Thus $(A, \mathcal{E}) \to (B, i\mathcal{E})$ is a pushout of a trivial cofibration. $\qquad\square$

Lemma 3.12. *Let $f : A \to B$ be a left marked anodyne map and $X \in \mathbf{SSet}$. Then the join $A \star X^\flat \to B \star X^\flat$ is left marked anodyne. Dually, a join $X^\flat \star A \to X^\flat \star B$ of X with a right marked anodyne map is right marked anodyne.*

Proof. We prove the left part. By [27, Lemma 4.10], the pushout-join of f with $\emptyset \to X^\flat$ is left marked anodyne. Unraveling the definition, the pushout-join is given by

$$
B \coprod_A A \star X^\flat \to B \star X^\flat.
$$

Our map can be factored as a composition

$$
A \star X^\flat \to B \coprod_A A \star X^\flat \to B \star X^\flat,
$$

and the left map is a pushout of the left marked anodyne map $A \to B$. $\qquad\square$

Lemma 3.13. *Let $\mathcal{D}$ be a direct category and $\{\mathcal{D}_\beta\}$ denote a good filtration. Then there are zig-zags of weak equivalences in $\mathbf{SSet}_+/\mathcal{D}$ for the coCartesian model structure:*

$$
Lat(x)^\flat \longrightarrow Lat(x)^\flat \times [1]^\sharp \longleftarrow (Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} Lat(x)_x^\flat,
$$

$$
Lat(x)^\flat \star \{x\} \longrightarrow (Lat(x)^\flat \times [1]^\sharp) \star \{x\} \longleftarrow (Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} (Lat(x)_x^\flat \star \{x\}).
$$

Here the left maps are induced by the inclusion $\{0\} \subset [1]^\sharp$ and $Lat(x)_x^\flat$ denotes the category $Lat(x)^\flat$ together with a constant functor to $\mathcal{D}$ of value x.

Proof. Denote by M either $\{x\}$ or the empty simplicial set. The observations of Lemmas 3.9 and 3.12 instantly imply that $\mathscr{L}(x)^\flat \star M^\flat \to (Lat(x)^\flat \times [1]^\sharp) \star M^\flat$ is left marked anodyne, and hence a coCartesian equivalence.

The remaining map is a pushout-join, without any markings, of $\emptyset \subset M$ with $Lat(x) \subset Lat(x) \times [1]$, with the latter map induced by the inclusion $\{1\} \subset [1]$, which is right anodyne. The stability properties of right anodyne maps, [18, Corollary 2.1.2.7], and [18, Lemma 2.1.2.3] again imply that the resulting pushout-join is inner anodyne. Lemma 3.11 then allows to conclude that

$$(Lat(x)^\flat \times [1]^\sharp) \star M^\flat \longleftarrow (Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} (Lat(x)_x^\flat \star M)$$

is a coCartesian eqiuvalence. $\qquad\square$

Proposition 3.14. *Let $\mathcal{X} \to \mathcal{D}$ be a fibrewise cocomplete coCartesian fibration over a direct Reedy category $\mathcal{D}$. Let $\mathcal{D}_\beta$ denote a good filtration of $\mathcal{D}$, so that $\mathcal{D}_\beta$ is obtained from $\mathcal{D}_{<\beta}$ by adding an object $x \in \mathcal{D}$. Then there is a Cartesian square in the quasicategory $\mathbf{Cat}_\infty$*

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{D}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{X}(x)) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{D}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathcal{X}(x)
\end{array}
$$

where

1. *the left vertical arrow is induced by the restriction along $\mathcal{D}_{<\beta} \to \mathcal{D}_\beta$,*

2. *the right vertical arrow is induced by the inclusion $[0] \to [1]$ skipping $1 \in [1]$,*

3. *the bottom horizontal arrow sends $S \in \mathsf{Sect}(\mathcal{D}_{<\beta}, \mathcal{X})$ to $\varinjlim_{Lat(x)} L_x S$. Here $L_x S$ is a x-left restriction of S,*

4. *the upper horizontal arrow sends $S \in \mathsf{Sect}(\mathcal{D}_\beta, \mathcal{X})$ to $\varinjlim_{Lat(x)} L_x S \to S(x)$.*

Proof. Proposition 3.3 implies that the square

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{D}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Sect}(Lat(x) \star \{x\}, \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{D}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Sect}(Lat(x), \mathcal{X})
\end{array}
$$

is Cartesian in $\mathbf{Cat}_\infty$. Lemma 3.13 then implies the existence of the following pullback square in $\mathbf{Cat}_\infty$:

$$
\begin{array}{ccc}
\mathsf{Sect}(Lat(x) \star \{x\}, \mathcal{X}) & \longrightarrow & \mathsf{Sect}((Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} (Lat(x)_x^\flat \star \{x\}), \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(Lat(x), \mathcal{X}) & \longrightarrow & \mathsf{Sect}((Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} Lat(x)_x^\flat, \mathcal{X}).
\end{array}
$$

For the right arrow, the notation $\mathsf{Sect}(-, \mathcal{X})$ means $\mathrm{Map}^\flat_{\mathcal{D}_\beta}(-, \mathcal{X}^\sharp)$. We notice that there is the following pullback square:

$$
\begin{array}{ccc}
\mathsf{Sect}((Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)^\flat_x} (Lat(x)^\flat_x \star \{x\}), \mathcal{X}) & \longrightarrow & \mathsf{Fun}(Lat(x) \star \{x\}, \mathcal{X}(x)) \\
\downarrow & & \downarrow \\
\mathsf{Sect}((Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)^\flat_x} Lat(x)^\flat_x, \mathcal{X}) & \longrightarrow & \mathsf{Fun}(Lat(x), \mathcal{X}(x))
\end{array}
$$

which is induced by applying $\mathsf{Sect}(-, \mathcal{X})$ to a (homotopy) pushout square of marked simplicial sets over $\mathcal{D}$. Combining all the squares with Proposition 3.5, we finish the proof. $\qquad\square$

To continue, we need to specify which fibrations we are going to consider over a general Reedy category $\mathcal{R}$. The higher-categorical generality that we choose to work with in this paper is motivated by Lemma 1.41.

Definition 3.15. Let $\mathcal{R}$ be a Reedy category. A *left Reedy fibration* is a coCartesian fibration of quasicategories $\mathcal{X} \to \mathcal{R}$ that is also a Cartesian (equivalently [18, Corollary 5.2.2.4] locally Cartesian) fibration over the subcategory $\mathcal{R}_-$. The notion of a right Reedy fibration is given dually.

Henceforth we shall stick with a left Reedy fibration $\mathcal{X} \to \mathcal{R}$. The corresponding results for a right Reedy fibration can be obtained by dualisation.

Lemma 3.16. *Let $\mathcal{R}$ be a Reedy category and $\{\mathcal{R}_\beta\}$ denote a good filtration as before. Let M denote either $\mathscr{M}(x)$ or $x \star Mat(x)$. Then there is a zig-zag of weak equivalences in $\mathbf{SSet}_+/\mathcal{R}_\beta$ for the coCartesian model structure:*

$$
\mathscr{L}(x)^\flat \star M^\flat \longrightarrow (\mathscr{L}(x)^\flat \times [1]^\sharp) \star M^\flat \longleftarrow (Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)^\flat_x} (Lat(x)^\flat_x \star M).
$$

Here the left map is induced by the inclusion $\{0\} \subset [1]^\sharp$ and $Lat(x)^\flat_x$ denotes the category $Lat(x)^\flat$ together with a constant functor to $\mathcal{R}$ of value x. A dual result can be formulated for $Mat(x)$ and the Cartesian model structure.

Proof. Verbatim Lemma 3.13. $\qquad\square$

Proposition 3.17. *Let $\mathcal{X} \to \mathcal{R}$ be a fibrewise bicomplete left Reedy fibration. Let $\mathcal{R}_\beta$ be a good filtration of $\mathcal{R}$, so that $\mathcal{R}_\beta$ is obtained from $\mathcal{R}_{<\beta}$ by adding an object $x \in \mathcal{R}$. Then there is a Cartesian square in the quasicategory $\mathbf{Cat}_\infty$*

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([2], \mathcal{X}(x)) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{X}(x))
\end{array}
\tag{3.6}
$$

54

where

1. *the left vertical arrow is induced by the restriction along $\mathcal{R}_{<\beta} \to \mathcal{R}_{\beta}$,*

2. *the right vertical arrow is induced by the inclusion $[1] \to [2]$ skipping $1 \in [2]$,*

3. *the bottom horizontal arrow sends $S \in \mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{X})$ to $\varinjlim_{Lat(x)} L_x S \to \varprojlim_{Mat(x)} R_x S$. Here $L_x S$ and $R_x S$ are x-left and right restrictions of S,*

4. *the upper horizontal arrow sends $S \in \mathsf{Sect}(\mathcal{R}_{\beta}, \mathcal{X})$ to*

$$
\begin{array}{ccc}
 & S(x) & \\
 & \nearrow \qquad \searrow & \\
\varinjlim_{Lat(x)} L_x S & \longrightarrow & \varprojlim_{Mat(x)} R_x S.
\end{array}
$$

Moreover,

i. *For each ordinal β, the $\mathbf{Cat}_{\infty}$-limit of $\{\mathsf{Sect}(\mathcal{R}_{\gamma}, \mathcal{X})\}_{\gamma < \beta}$ is equivalent, via the evident map, to $\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{X})$.*

ii. *The $\mathbf{Cat}_{\infty}$-limit of $\{\mathsf{Sect}(\mathcal{R}_{\beta}, \mathcal{X})\}_{\beta}$ is equivalent, via the evident map, to $\mathsf{Sect}(\mathcal{R}, \mathcal{X})$.*

Proof. The last two statements, (i.) and (ii.), follow immediately from the properties of the filtration $\{\mathcal{R}_{\beta}\}$. For the rest, use again Proposition 3.3, and from the diagram

$$
\begin{array}{ccccc}
Lat(x) \star Mat(x) & \longrightarrow & \mathcal{R}_{<\beta} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
Lat(x) \star \{x\} \star Mat(x) & \longrightarrow & \mathcal{R}_{\beta} & \longrightarrow & \mathcal{R},
\end{array}
$$

get a Cartesian diagram in $\mathbf{Cat}_{\infty}$

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_{\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Sect}(Lat(x) \star \{x\} \star Mat(x), \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Sect}(Lat(x) \star Mat(x), \mathcal{X}).
\end{array}
$$

Lemma 3.16 implies that sections over $Lat(x) \star Mat(x)$ can be replaced with sections over $(Lat(x) \times [1]) \coprod_{Lat(x)_x} (Lat(x)_x \star Mat(x))$ which are coCartesian along certain edges coming from $[1]$ in $Lat(x) \times [1]$. We can depict this by asserting the existence of the $\mathbf{Cat}_{\infty}$-Cartesian diagram:

$$
\begin{array}{ccc}
\mathsf{Sect}(Lat(x) \star \{x\} \star Mat(x), \mathcal{X}) & \longrightarrow & \mathsf{Sect}((Lat(x)^{\flat} \times [1]^{\sharp}) \coprod_{Lat(x)_x^{\flat}} (Lat(x)_x^{\flat} \star \{x\} \star Mat(x)^{\flat}), \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(Lat(x) \star Mat(x), \mathcal{X}) & \longrightarrow & \mathsf{Sect}((Lat(x)^{\flat} \times [1]^{\sharp}) \coprod_{Lat(x)_x^{\flat}} (Lat(x)_x^{\flat} \star Mat(x)^{\flat}), \mathcal{X})
\end{array}
$$

obtained from a zig-zag in $\mathbf{SSet}^{[1]}$ of length two. For the right arrow, we notice that there is the following pullback square:

$$\mathsf{Sect}((Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} (Lat(x)_x^\flat \star \{x\} \star Mat(x)^\flat), \mathfrak{X}) \longrightarrow \mathsf{Sect}(Lat(x)_x \star \{x\} \star Mat(x), \mathfrak{X})$$

$$\mathsf{Sect}((Lat(x)^\flat \times [1]^\sharp) \coprod_{Lat(x)_x^\flat} (Lat(x)_x^\flat \star Mat(x)^\flat), \mathfrak{X}) \longrightarrow \mathsf{Sect}(Lat(x)_x \star Mat(x), \mathfrak{X}).$$

Note that the pullback of $\mathfrak{X} \to \mathcal{R}$ to $Lat(x)_x \star \{x\} \star Mat(x)$ is a genuine Cartesian fibration. A similar argument concerning the replacement of $Mat(x)$ supplies us with the following pullback square in $\mathbf{Cat}_\infty$:

$$\mathsf{Sect}(Lat(x)_x \star \{x\} \star Mat(x), \mathfrak{X}) \longrightarrow \mathsf{Fun}(Lat(x) \star \{x\} \star Mat(x), \mathfrak{X}(x))$$

$$\mathsf{Sect}(Lat(x)_x \star Mat(x), \mathfrak{X}) \longrightarrow \mathsf{Fun}(Lat(x) \star Mat(x), \mathfrak{X}(x)).$$

Proposition 3.5 then concludes the proof. $\qquad\square$

We conclude our discussion by outlining the functoriality of the diagram (*3.6*) that will be useful for the purposes of the comparison.

Definition 3.18. Let $\mathcal{R}$ be a Reedy category and $F : \mathfrak{X} \to \mathfrak{Y}$ be an infinity-functor between fibrewise bicomplete left Reedy fibrations over $\mathcal{R}$. Let $\mathcal{R}_\beta$ be the step of a good filtration that adds x. A section $X : \mathcal{R}_{<\beta} \to \mathfrak{X}$ is called (F, x)-*compatible* if the following holds:

i. For each $y \in \mathcal{R}_{<\beta}$, the functor F preserves coCartesian arrows starting with $X(y)$ and Cartesian arrows over $\mathcal{R}_-$ ending with $X(y)$,

ii. The cone $Lat(x)^\triangleright \to \mathfrak{X}(x)$ obtained by taking a x-left restriction of X and then taking the colimit, remains a colimit cone in $\mathfrak{Y}(x)$ cone after postcomposing with F,

iii. The cone $Mat(x)^\triangleleft \to \mathfrak{X}(x)$ obtained by taking a x-right restriction of X and then taking the limit, remains a limit cone in $\mathfrak{Y}(x)$ after postcomposing with F.

A general section $\mathcal{R} \to \mathfrak{X}$ is (F, x)-compatible iff its restriction to $\mathcal{R}_{<\beta}$ is such. Denote by $\mathsf{Sect}_{(F,x)}(\mathcal{R}_{<\beta}, \mathfrak{X})$, $\mathsf{Sect}_{(F,x)}(\mathcal{R}_\beta, \mathfrak{X})$ and $\mathsf{Sect}_{(F,x)}(\mathcal{R}, \mathfrak{X})$ the full subcategories spanned by (F, x)-compatible functors.

An obvious compatibility is the following:

Lemma 3.19. *Let $F : \mathfrak{X} \to \mathfrak{Y}$ be an infinity-functor between fibrewise bicomplete left Reedy fibrations over $\mathcal{R}$. Then F preserves x-left restrictions and x-right restrictions of (F, x)-compatible sections.*

Proof. Immediate using (i.) of Definition 3.18. □

Proposition 3.20. *Let $\mathcal{R}$ be a Reedy category and and $F : \mathcal{X} \to \mathcal{Y}$ be an infinity-functor between fibrewise bicomplete left Reedy fibrations over $\mathcal{R}$. Then, if $\mathcal{R}_\beta$ is obtained from $\mathcal{R}_{<\beta}$ by adding x, the diagram (3.6) of Proposition 3.17 induces the diagram $\mathcal{D}ia(\mathcal{R}_\beta, F)$:*

$$
\begin{array}{ccc}
\mathsf{Sect}_{(F,x)}(\mathcal{R}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([2], \mathcal{X}(x)) \\
\downarrow & & \downarrow \\
\mathsf{Sect}_{(F,x)}(\mathcal{R}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{X}(x)).
\end{array}
\tag{3.7}
$$

Moreover, post-composing with F induces a natural transformation of diagrams

$$F_* : \mathcal{D}ia(\mathcal{R}_\beta, F) \to \mathcal{D}ia(\mathcal{R}_\beta, Id_\mathcal{Y})$$

with $\mathcal{D}ia(\mathcal{R}_\beta, Id_\mathcal{Y})$ denoting the diagram (3.6) for $\mathcal{Y} \to \mathcal{R}$.

Proof. The definition of compatibility implies that the following square is pullback in **SSet**:

$$
\begin{array}{ccc}
\mathsf{Sect}_{(F,x)}(\mathcal{R}_\beta, \mathcal{X}) & \longrightarrow & \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Sect}_{(F,x)}(\mathcal{R}_{<\beta}, \mathcal{X}) & \longrightarrow & \mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{X});
\end{array}
$$

the right functor is moreover a categorical fibration, since $\mathcal{R}_{<\beta} \subset \mathcal{R}_\beta$ is an inclusion (and it becomes a cofibration in the coCartesian model structure on $\mathbf{SSet}_+/\mathcal{R}$). This diagram is hence a homotopy pullback, so the existence of (*3.7*) is clear. To see the functoriality, write the diagram (*3.6*) as a zig-zag. A close inspection shows that $F : \mathcal{X} \to \mathcal{Y}$ induces a map between the zig-zags representing (*3.7*) and (*3.6*): the nontrivial moments are covered by Lemma 3.19 and observing that the restriction of the functor (cf notation for the diagram (*3.4*))

$$\mathsf{Fun}'(Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft, \mathcal{X}(x)) \to \mathsf{Fun}(Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft, \mathcal{Y}(x))$$

to the diagrams $Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft \to \mathcal{X}(x)$ which come from (F, x)-compatible sections, factors through $\mathsf{Fun}'(Lat(x)^\triangleright \star \{x\} \star Mat(x)^\triangleleft, \mathcal{Y}(x))$. □

3.2 Families of relative categories

Definition 3.21. Let $(\mathcal{X}, \mathcal{W})$ be a relative category and $p : \mathcal{X} \to \mathcal{C}$ a functor sending $\mathcal{W}$ to $\mathcal{C}$-isomorphisms. An *infinity-localisation* of p is the factoring of it as an infinity-localisation $\mathcal{X} \to L\mathcal{X}$ along $\mathcal{W}$ followed by a categorical fibration $L\mathcal{X} \to \mathcal{C}$ of quasicategories.

An infinity-localisation of p is *universal* iff for any functor $F : \mathcal{D} \to \mathcal{C}$, the induced map $LF^*\mathcal{X} \to F^*L\mathcal{X}$ is a categorical equivalence.

Using the model structure on marked simplicial sets *over the point*, we can always (and even functorially) factor $(\mathcal{X}, \mathcal{W}) \to (\mathcal{C}, Iso_{\mathcal{C}}) = \mathcal{C}^\natural$ as a trivial cofibration followed by a fibration. This shows that the infinity-localisation of functors always exists.

Let $\mathcal{E} \to \mathcal{C}$ be an opfibration. Assume that each fibre $\mathcal{E}(c)$ is equipped with weak equivalences $\mathcal{W}(c)$ that are preserved by the transition functors: if in a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\sim \downarrow & & \downarrow \\
Y & \longrightarrow & Y'
\end{array}
$$

we have horizontal arrows opcartesian and $X \to Y \in \mathcal{W}(c)$, then $X' \to Y' \in \mathcal{W}(c')$. The data of $\mathcal{W} = Iso_{\mathcal{E}} \cup (\cup_c \mathcal{W}(c))$ provides (the nerve of) $\mathcal{E}$ with the structure of a marked simplicial set. Choose an infinity-localisation $(\mathcal{E}, \mathcal{W}) \overset{\sim}{\hookrightarrow} L\mathcal{E} \twoheadrightarrow \mathcal{C}^\natural$ of the functor $\mathcal{E} \to \mathcal{C}$.

Since $L\mathcal{E}$ is fibrant in $\mathbf{SSet}_+$, it is a localisation of $\mathcal{E}$. One furthermore has [20, Proposition 5.2.3] that $L\mathcal{E} \to \mathcal{C}$ is a coCartesian fibration classified by the functor $c \mapsto L\mathcal{E}(c)$, and the infinity-functor $\mathcal{E} \to L\mathcal{E}$ preserves coCartesian arrows. Taking into account the naturality of the Grothendieck construction [20, Remark 3.1.13], we also have that for any functor $F : \mathcal{D} \to \mathcal{C}$, the natural morphism $LF^*\mathcal{E} \to F^*L\mathcal{E}$ is a coCartesian equivalence over $\mathcal{C}$. The localisation of $\mathcal{E} \to \mathcal{C}$ is thus universal.

Remark 3.22. A result of Hinich [11, 2.1.4] is similar to that of Mazel-Gee [20, Proposition 5.2.3] with a distinction: aside from inverting also the maps in the base, the fibration localiser $\mathcal{W}$ is assumed to be saturated. The way we construct $\mathcal{W} = Iso_{\mathcal{E}} \cup (\cup_c \mathcal{W}(c))$ does not usually produce a saturated localiser, however in the Reedy setting this actually happens. In many cases, $\mathcal{W}$ saturates to a bigger class of weak equivalences: those maps $X \to Y$ in $\mathcal{M}$ which projection $f : x \to y$ is an isomorphism and the induced map $f_! X \to Y$ is in $\mathcal{W}(y)$.

If we a dealing with, say, a Quillen presheaf $\mathcal{N} \to \mathcal{C}$, then the transition adjunctions

$$
\mathcal{N}(c) \rightleftarrows \mathcal{N}(c')
$$

do not in general preserve weak equivalences. Instead, they preserve weak equivalences only if restricted to the subcategories of cofibrant or fibrant objects, depending on the direction. To study what happens if we localise $\mathcal{N}$ in this case, let us first introduce some tools.

Consider a relative category $(\mathcal{M}, \mathcal{W})$. Given a subcategory $\mathcal{M}_Q \subset \mathcal{M}$ whose objects we shall typically denote QX, for any composable sequence $X_0 \to ... \to X_n$ of morphisms in $\mathcal{M}$, denote by

$$
\mathcal{M}_Q/^{\mathcal{W}}(X_0 \to ... \to X_n)
$$

the category whose objects are diagrams (with the bottom row always fixed)

$$
\begin{array}{ccc}
QX_0 & \longrightarrow ... \longrightarrow & QX_n \\
\sim \downarrow & & \sim \downarrow \\
X_0 & \longrightarrow ... \longrightarrow & X_n
\end{array}
$$

with vertical arrows in $\mathcal{W}$, and morphisms being weak equivalences between the top rows. One can dually define an undercategorical analogue that we shall denote $(X_0 \to ... \to X_n)\backslash^{\mathcal{W}}\mathcal{M}_Q$.

Definition 3.23. Let $(\mathcal{M}, \mathcal{W})$ be a relative category.

1. A *left approximation* is a full subcategory $\mathcal{M}_Q \subset \mathcal{M}$ such that for each sequence $X_0 \to ... \to X_n$, the category $\mathcal{M}_Q/^{\mathcal{W}}(X_0 \to ... \to X_n)$ is contractible.

2. A *right approximation* is a full subcategory $\mathcal{M}_R \subset \mathcal{M}$ such that for each sequence $X_0 \to ... \to X_n$, the category $(X_0 \to ... \to X_n)\backslash^{\mathcal{W}}\mathcal{M}_R$ is contractible.

The following lemma justifies the choice of terminology.

Lemma 3.24. *Let $\mathcal{M}_Q \subset \mathcal{M}$ be a left approximation of a relative category $(\mathcal{M}, \mathcal{W})$. Then*

$$(\mathcal{M}_Q, \mathcal{W}|_{\mathcal{M}_Q}) \to (\mathcal{M}, \mathcal{W})$$

is a weak equivalence of relative categories. There is a similar result that concerns right approximations.

Recall [5] [2, 2.2] that a functor $F : \mathcal{M} \to \mathcal{M}'$ of relative categories is a *weak equivalence*, or simply an equivalence, if the induced map of the infinity-localisations $LF : L\mathcal{M} \to L\mathcal{M}'$ is an equivalence of quasicategories. This condition is equivalent to that, for each $n \geq 0$, the functor

$$F_* : \mathsf{Fun}_{\mathcal{W}}([n], \mathcal{M}) \longrightarrow \mathsf{Fun}_{\mathcal{W}'}([n], \mathcal{M}')$$

is a weak equivalence of the associated nerves. Here $\mathsf{Fun}_{\mathcal{W}}([n], \mathcal{M})$ denotes the subcategory of pointwise weak equivalences in $\mathsf{Fun}([n], \mathcal{M})$ and similarly for $\mathcal{M}'$.

Proof. Denote by $\tilde{\mathcal{M}}$ the full subcategory of $\mathsf{Fun}([1], \mathcal{M})$ consisting of weak equivalences $QX \overset{\sim}{\to} X$ where QX belongs to $\mathcal{M}_Q$. We have then a natural factorisation $\mathcal{M}_Q \to \tilde{\mathcal{M}} \to \mathcal{M}$. It is easy to see that $\mathcal{M}_Q \to \tilde{\mathcal{M}}$ is a weak equivalence of relative categories, since it admits a homotopy inverse.

We observe that for each n, the functor $\mathsf{Fun}_{\mathcal{W}}([n], \tilde{\mathcal{M}}) \to \mathsf{Fun}_{\mathcal{W}}([n], \mathcal{M})$ is an opfibration, hence by Quillen Theorem A it is sufficient to prove that its fibres are contractible. However, the fibre of this functor over $X_0 \to ... \to X_n$ is exactly $\mathcal{M}_Q/^{\mathcal{W}}(X_0 \to ... \to X_n)$. $\qquad\square$

As the choice of notation suggests, the fundamental examples [11, 1.3.7] [2, Lemma 2.4.8] of left approximations are given by the subcategories $\mathcal{M}_c \subset \mathcal{M}$ of cofibrant objects in a model category $\mathcal{M}$. Fibrant objects, dually, serve as right approximations. The identity functor is of course both left and right approximation. We will have further examples below.

In the case of (co)fibrant objects in a model category, a stronger approximation property holds.

Definition 3.25. Let $(\mathcal{M}, \mathcal{W})$ be a relative category. A *strong left approximation* is a full subcategory $\mathcal{M}_Q \subset \mathcal{M}$ such that

i. for each $X \in \mathcal{M}$, the category $\mathcal{M}_Q/^{\mathcal{W}}X$ is contractible, and

ii. for each $X \to Y \in \mathcal{M}^{[1]}$, the projection $\mathcal{M}_Q/^{\mathcal{W}}(X \to Y) \to \mathcal{M}_Q/^{\mathcal{W}}X$ has contractible fibres.

the definition of a strong right approximation is given dually.

Lemma 3.26. *Any strong left approximation is a left approximation, and dually, any strong right approximation is a right approximation.*

Proof. The projection functor

$$\mathcal{M}_Q/^{\mathcal{W}}(X_0 \to ... \to X_n) \to \mathcal{M}_Q/^{\mathcal{W}}(X_0 \to ... \to X_{n-1})$$

is a Grothendieck fibration whose fibres are the same as the fibres of the projection

$$\mathcal{M}_Q/^{\mathcal{W}}(X_{n-1} \to X_n) \to \mathcal{M}_Q/^{\mathcal{W}}X_{n-1}.$$

Quillen Theorem A and induction imply the result. $\qquad\square$

Lemma 3.27. *Let $\mathcal{M}$ be a model category. The inclusion $\mathcal{M}_c \subset \mathcal{M}$ of cofibrant objects is a strong left approximation. The inclusion $\mathcal{M}_f \subset \mathcal{M}$ of fibrant objects is a strong right approximation.*

Proof. We prove the left part. The contractibility of $\mathcal{M}_c/^{\mathcal{W}}X$ is already established [2, Lemma 2.4.8]. We now have to prove that for each $X \to Y$ and a weak equivalence $QX \overset{\sim}{\to} X$ with QX cofibrant, the category of fillers of the upper right corner in the square

$$
\begin{array}{ccc}
QX & \dashrightarrow & QY \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
X & \longrightarrow & Y
\end{array}
$$

is contractible. Note that this will follow if we prove that for any map $f : QX \to Y$ with QX cofibrant, the category $Fact(f)$ of factorisations

$$
\begin{array}{ccc}
 & QY & \\
\nearrow & & \searrow{\scriptstyle\sim} \\
QX & \xrightarrow{\quad f \quad} & Y
\end{array}
\qquad (3.8)
$$

with QY cofibrant and $QY \to Y$ a weak equivalence, is contractible.

Denote by $Fact_c(f) \subset Fact(f)$ the subcategory consisting of all the factorisations in which $QX \to QY$ is a cofibration. The comma-fibre of this functor over an object (3.8) is the category of cofibrant replacements of (3.8) in the model category $f\backslash(\mathcal{M}/Y)$. Hence

$Fact_c(f) \rightarrow Fact(f)$ is a homotopy equivalence. The category $Fact_c(f)$ is however contractible: an object

$$
\begin{array}{ccc}
 & QY & \\
 \nearrow & & \searrow{\scriptstyle\sim} \\
QX & \xrightarrow{f} & Y
\end{array}
$$

is a cofibrant replacement of f in $QX\backslash\mathcal{M}$. $\qquad\square$

Left and right approximations interact reasonably well in families. The following propositions will be sufficient for our needs. First, consider a typical situation when one has a family of model categories and functors that preserve cofibrations and trivial cofibrations. Such functors do not preserve weak equivalences on the nose, but we can always limit our attention to the cofibrant objects:

Lemma 3.28. *Let $p : \mathcal{M} \rightarrow \mathcal{C}$ be an opfibration such that each fibre $\mathcal{M}(c)$ is equipped with weak equivalences $\mathcal{W}(c)$ and a strong left approximation $\mathcal{M}_Q(c)$, subject to the condition that given a diagram*

$$
\begin{array}{ccc}
QX & \longrightarrow & X' \\
{\scriptstyle\sim}\downarrow & & \downarrow \\
QY & \longrightarrow & Y'
\end{array}
$$

with left vertical arrow a fibrewise weak equivalence, horizontal arrows opcartesian, and $QX, QY \in \mathcal{M}_Q(c)$ for some c, we have that $X' \rightarrow Y'$ is a weak equivalence in $\mathcal{M}_Q(c')$.

Denote by $\mathcal{M}_Q \subset \mathcal{M}$ the subcategory spanned by all objects from all $\mathcal{M}_Q(c)$. Then $\mathcal{M}_Q \rightarrow \mathcal{C}$ is an opfibration whose transition functors preserve weak equivalences. Moreover, the inclusion $\mathcal{M}_Q \subset \mathcal{M}$ is a strong left approximation, where we endow $\mathcal{M}$ with the totality of fibrewise weak equivalences $\mathcal{W} = \cup_c\mathcal{W}(c)$.

Proof. The only nontrivial part is proving that $\mathcal{M}_Q \subset \mathcal{M}$ is a strong left approximation. For a single object X, the category $\mathcal{M}_Q/^{\mathcal{W}}X$ is the same as $\mathcal{M}_Q(x)/^{\mathcal{W}(x)}X$ for $pX = x$. It is hence contractible by assumption.

For $\alpha : X \rightarrow Y$ we study the fibres of the forgetful functor

$$
U : \mathcal{M}_Q/^{\mathcal{W}}(X \rightarrow Y) \rightarrow \mathcal{M}_Q/^{\mathcal{W}}X.
$$

Over $QX \xrightarrow{\sim} X$, the fibre is the category of fillers of the right upper corner in the following square:

$$
\begin{array}{ccc}
QX & \dashrightarrow & QY \\
{\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\vdots \\
X & \longrightarrow & Y;
\end{array}
$$

This is the same category as the category of fillers in the diagram

$$
\begin{array}{ccc}
QX & \dashrightarrow & QY \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle \sim}\big\downarrow \\
QX & \longrightarrow & Y
\end{array}
$$

for the composition $QX \to X \to Y$. The latter is the same as the fibre of

$$
U' : \mathcal{M}_Q(y)/^{\mathcal{W}}(p\alpha_! QX \to Y) \to \mathcal{M}_Q(y)/^{\mathcal{W}}p\alpha_! X
$$

over $p\alpha_! QX \to p\alpha_! QX$. $\qquad\square$

Given a relative category $(\mathfrak{X}, \mathcal{W})$, we say that $\mathcal{W}$ is saturated if any map that becomes invertible in $L\mathfrak{X}$, belongs to $\mathcal{W}$. Weak equivalences of model categories are saturated.

Corollary 3.29. *In the situation of Lemma 3.28, consider an infinity-localisation $\mathcal{M} \to L\mathcal{M} \to \mathcal{C}$ of $\mathcal{M} \to \mathcal{C}$. Then $L\mathcal{M} \to \mathcal{C}$ is a coCartesian fibration of quasicategories, and is a* universal *localisation of $\mathcal{M} \to \mathcal{C}$.*

Assume that each $\mathcal{W}(c)$ is saturated. Then a map $\alpha : X \to Y$ of $\mathcal{M}$ becomes coCartesian in $L\mathcal{M}$ iff for any $\mathcal{M}_Q$-approximation $QX \to X$, the induced map $p\alpha_! QX \to Y$ is a weak equivalence.

Note that the last sentence implies that the infinity-functor $\mathcal{M} \to L\mathcal{M}$ preserves coCartesian arrows with $\mathcal{M}_Q$-domain.

Proof. Since $\mathcal{M}_Q \subset \mathcal{M}$ is a weak equivalence of relative categories, one has $L\mathcal{M}_Q \cong L\mathcal{M}$. Hence $L\mathcal{M} \to \mathcal{C}$ is a categorical fibration that is equivalent to a coCartesian fibration, hence it is a coCartesian fibration itself [21, Corollary 3.4]. Moreover, for any $F : \mathcal{D} \to \mathcal{C}$, we have an equivalence $LF^* \mathcal{M}_Q \cong LF^* \mathcal{M}$, and since $L\mathcal{M}_Q \to \mathcal{C}$ is universal, same result follows for $L\mathcal{M} \to \mathcal{C}$.

Furthermore, the functor $L\mathcal{M}_Q \cong L\mathcal{M}$ preserves coCartesian arrows, since in this case being coCartesian coincides with the invariant definition [21, Theorem 3.3], and equivalences preserve invariantly defined coCartesian arrows. The infinity-functor $\mathcal{M}_Q \to L\mathcal{M}_Q$ preserves coCartesian arrows as well.

The last part of the corollary then follows from the observation that for $\alpha : X \to Y$ and a $\mathcal{M}_Q$-approximation $QX \xrightarrow{\sim} X$, we have the following diagram in $\mathcal{M}$:

$$
\begin{array}{ccc}
QX & \longrightarrow & p\alpha_! QX \\
{\scriptstyle \sim}\big\downarrow & & \big\downarrow \\
X & \xrightarrow{\ \ \alpha\ \ } & Y,
\end{array}
$$

the map α being $L\mathcal{M}$-coCartesian thus being equivalent to $p\alpha_! QX \to Y$ becoming invertible in $L\mathcal{M}$. $\qquad\square$

Our next result is a variation of [20, Theorem 5.3.1].

Lemma 3.30. *Let $(\mathcal{C}, \mathcal{W}_{\mathcal{C}})$ and $(\mathcal{D}, \mathcal{W}_{\mathcal{D}})$ be two relative categories and*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

an adjunction such that

> *i. there is a left approximation $\mathcal{C}_Q \subset \mathcal{C}$ such that F preserves weak equivalences between the objects of $\mathcal{C}_Q$,*
>
> *ii. there is a right approximation $\mathcal{D}_R \subset \mathcal{D}$ such that G preserves weak equivalences between the objects of $\mathcal{D}_R$.*

Let $\mathcal{M} \to [1]$ be the bifibration corresponding to the adjunction $F \dashv G$ that we endow with the weak equivalences $\mathcal{W} = \mathcal{W}_{\mathcal{C}} \cup \mathcal{W}_{\mathcal{D}}$. Denote by $\mathcal{M}_l$ the subcategory of $\mathcal{M}$ spanned by $\mathcal{C}_Q$ and $\mathcal{D}$, and by $\mathcal{M}_r$ the subcategory spanned by $\mathcal{C}$ and $\mathcal{D}_R$. Then the inclusions $\mathcal{M}_l \subset \mathcal{M} \supset \mathcal{M}_r$ are, respectively, a left and a right approximation. These approximations are strong whenever $\mathcal{C}_Q \subset \mathcal{C}$ and $\mathcal{D}_R \subset \mathcal{D}$ are.

Proof. We prove the left part. A typical string of arrows in $\mathcal{M}$ is given by

$$X_0 \to ... \to X_m \to Y_0 \to ... \to Y_n$$

with X_i in $\mathcal{C}$ and Y_j in $\mathcal{D}$. Formally, m and n can take value -1, so that no X or Y appear in the string. When this happens, we are reduced to proving that $\mathcal{C}_Q \subset \mathcal{C}$ or $\mathcal{D} = \mathcal{D}$ are left approximations, which is given.

For the middle case, consider the functor

$$\mathcal{M}_l/^{\mathcal{W}}(X_0 \to ... \to X_m \to Y_0 \to ... \to Y_n) \to \mathcal{C}_Q/^{\mathcal{W}_{\mathcal{C}}}(X_0 \to ... \to X_m)$$

this is a Grothendieck fibration whose fibres have final objects. We thus conclude what is needed. The treatment of the strong case is similar to the proof of Lemma 3.28. $\square$

Corollary 3.31. *In the situation of Lemma 3.30, consider an infinity-localisation $\mathcal{M} \to L\mathcal{M} \to [1]$. Then $L\mathcal{M} \to [1]$ is a biCartesian fibration that classifies an infinity-adjunction*

$$\mathbb{L}F : L\mathcal{C} \rightleftarrows L\mathcal{D} : \mathbb{R}G.$$

The value of $\mathbb{L}F$ at X is represented by FQX, and the value of $\mathbb{R}G$ at Y is represented by GRY.

The value of the unit of the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ at $X \in L\mathcal{C}$ is equivalent to the image of the following chain of $\mathcal{C}$-morphisms:

$$X \overset{\sim}{\leftarrow} QX \to GFQX \to G(RFQX)$$

with $QX \overset{\sim}{\to} X$ and $FQX \overset{\sim}{\to} RFQX$ being the approximations. The expression for the counit is given dually.

Proof. Lemma 3.30 allows us to conclude that $LM_l \cong LM \cong LM_r$, hence $LM \to [1]$ is a categorical fibration that is equivalent to both a Cartesian and a coCartesian fibration. It is also universal, so $L\mathcal{C} \cong LM(0)$ and $L\mathcal{D} \cong LM(1)$. Same analysis as in Corollary 3.29 permits us to conclude that $\alpha : X \to Y$ of M becomes coCartesian in LM if $p\alpha_! QX \to Y$ is a weak equivalence, and similarly for becoming Cartesian.

To see the unit statement, recall [18, Proposition 5.2.2.8]. Given a biCartesian fibration $\mathcal{B} \to [1]$ and $x \in \mathcal{B}(0)$, first choose a coCartesian arrow $x \to f_! x$ covering $0 \to 1$, and then a Cartesian arrow $f^* f_! x \to f_! x$ covering the same arrow. The induced arrow $x \to f^* f_! x$ is (equivalent to) the value of the unit of the adjunction $f_! \dashv f^*$ at x.

We now observe that for $QX \in \mathcal{C}_Q$, the composition $QX \to FQX \xrightarrow{\sim} RFQX$ becomes coCartesian in LM. Since Cartesian maps with $\mathcal{D}_R$-codomains remain Cartesian, the resulting map $QX \to G(RFQX)$ represents the unit of the infinity-adjunction $\mathbb{L}F \dashv \mathbb{R}G$ evaluated at QX. However, it admits a factorisation $QX \to GFQX \to G(RFQX)$ in $\mathcal{C}$. $\qquad\square$

We conclude this subsection with a couple of lemmas that will permit us to conclude that the higher-categorical Reedy induction holds for the localisations of model categories of sections. For a relative category (M, W), denote by $L^H M$ its Dwyer-Kan "hammock" localisation.

Lemma 3.32. *Let*

$$
\begin{array}{ccc}
(M_1, W_1) & \xrightarrow{F} & (M_2, W_2) \\
{\scriptstyle G}\downarrow & & \downarrow{\scriptstyle K} \\
(M_3, W_3) & \xrightarrow[H]{} & (M_4, W_4)
\end{array}
$$

be a diagram of relative categories and equivalence preserving functors, such that

  i. The induced square

$$
\begin{array}{ccc}
\mathrm{Ob}\, M_1 & \longrightarrow & \mathrm{Ob}\, M_2 \\
\downarrow & & \downarrow \\
\mathrm{Ob}\, M_3 & \longrightarrow & \mathrm{Ob}\, M_4
\end{array}
$$

  is a pullback.

  ii. for each $X, Y \in M_1$, the induced square of simplicial sets

$$
\begin{array}{ccc}
L^H M_1(X, Y) & \longrightarrow & L^H M_2(FX, FY) \\
\downarrow & & \downarrow \\
L^H M_3(GX, GY) & \longrightarrow & L^H M_4(KFX, KFY)
\end{array}
$$

  is a homotopy pullback.

iii. The functor $\operatorname{Ho} K : \operatorname{Ho} \mathcal{M}_2 \to \operatorname{Ho} \mathcal{M}_4$ *is an isofibration.*

Then the induced square

$$
\begin{array}{ccc}
L^H \mathcal{M}_1 & \longrightarrow & L^H \mathcal{M}_2 \\
\downarrow & & \downarrow \\
L^H \mathcal{M}_3 & \longrightarrow & L^H \mathcal{M}_4
\end{array}
\qquad (3.9)
$$

is a homotopy pullback of simplicial categories.

Proof. We use the key observation of [14, Lemma 3.1.11] asserting that a functor $F : \mathcal{C} \to \mathcal{D}$ of simplicial categories can be factored as a weak equivalence and an isomorphism on object-sets followed by a map that is a fibration on mapping spaces. Using this observation we replace the diagram (3.9) by a weakly equivalent diagram

$$
\begin{array}{ccc}
L^H \mathcal{M}_1 & \longrightarrow & (L^H \mathcal{M}_2)' \\
\downarrow & & \downarrow \\
L^H \mathcal{M}_3 & \longrightarrow & (L^H \mathcal{M}_4)'
\end{array}
$$

where $(L^H \mathcal{M}_2)' \to (L^H \mathcal{M}_4)'$ is now a genuine fibration of simplicial categories. It is still true that the diagram of object-sets is a pullback, and an analogue of (ii.) holds for this diagram as well. It thus follows that

$$
L^H \mathcal{M}_1 \to L^H \mathcal{M}_3 \times_{(L^H \mathcal{M}_4)'} (L^H \mathcal{M}_2)'
$$

is bijective on objects and homotopically fully faithful. Since the model structure on simplicial categories is right proper, our proof is complete. $\qquad \square$

Lemma 3.33. *Let α be an ordinal and assume given, for each $\beta < \alpha$, a simplicial category A_β, and for each $\beta' < \beta$, a simplicial functor $A_\beta \to A_{\beta'}$ rendering the assignment $\beta \mapsto A_\beta$ functorial. Assume furthermore given, for each β, a simplicial functor $A_\beta \to A_{<\beta}$, and for each $\gamma < \beta$, a commutative diagram*

$$
\begin{array}{ccc}
& A_\beta & \\
\swarrow & & \searrow \\
A_{<\beta} & \longrightarrow & A_\gamma
\end{array}
$$

exhibiting $A_{<\beta}$ as a cone of $\{A_\gamma\}_{\gamma < \beta}$.

We require that the following holds.

0. For each successor ordinal the functor $A_{\beta+1} \to A_{<\beta+1}$ is equal to the functor $A_{\beta+1} \to A_\beta$,

1. For each ordinal β, the functor $A_\beta \to A_{<\beta}$ is an isofibration after applying Ho,

2. *For each ordinal β, the functors $A_{<\beta} \to A_\gamma$ exhibit $\mathrm{Ob}\, A_{<\beta}$ as a limit of $\{\mathrm{Ob}\, A_\beta\}_{\gamma<\beta}$,*

3. *For each ordinal β and $x, y \in A_{<\beta}$, denote by x_γ, y_γ their image in A_γ. Then the maps $A_{<\beta} \to A_\gamma$ exhibit the simplicial set $A_{<\beta}(x, y)$ as a homotopy limit of $\{A(x_\gamma, y_\gamma)\}_{\gamma<\beta}$*

Let A be a simplicial category and $A \to A_\beta$ be a compatible family of functors such that

i. *The maps $A \to A_\beta$ exhibit $\mathrm{Ob}\, A$ as a limit of $\mathrm{Ob}\, A_\beta$,*

ii. *For each $x, y \in A$, denote by x_β, y_β their image in A_β. Then the maps $A \to A_\beta$ exhibit the simplicial set $A(x, y)$ as a homotopy limit of $A(x_\beta, y_\beta)$.*

Then for any ordinal β, $A_{<\beta}$ is the homotopy limit of $\{A_\gamma\}_{\gamma<\beta}$, and A is the homotopy limit of $\{A_\beta\}_{\beta<\alpha}$.

Note that due to (0.), the conditions (2.) and (3.) are nontrivial only for a limit ordinal β.

Proof. Use transfinite induction to construct a new diagram of fibrations between fibrant simplicial categories that we shall denote $\{B_\beta\}$, together with suitable maps $B_\beta \to B_{<\beta}$. For $\beta = 0$, the map $A_0 \to B_0$ is the weak equivalence that is an identity on objects followed by a fibration $B_0 \to [0]$. For the inductive step, set $B_{<\beta} = \varprojlim_{\gamma<\beta} B_\gamma$. An elementary lifting argument shows that for each $\gamma < \beta$, the projections $B_{<\beta} \to B_\gamma$ are fibrations. Moreover $B_{<\beta}$ is the limit of a diagram of fibrations between fibrant simplicial categories, hence a homotopy limit as well. Conditions (2.) and (3.) imply that $A_{<\beta} \to B_{<\beta}$ is bijective on objects and weakly fully faithful, hence an equivalence of simplicial categories.

We now factor the composition $A_\beta \to A_{<\beta} \to B_{<\beta}$ as an object-bijective weak equivalence $A_\beta \to B_\beta$ followed by a local fibration $B_\beta \to B_{<\beta}$. Since bijective-on-objects equivalences are isofibrations, the simplicial functor $B_\beta \to B_{<\beta}$ will be a Ho-isofibration and hence a fibration as well. The induction is thus complete and the limit of $\{B_\beta\}_{\beta<\alpha}$ is also the homotopy limit.

To complete the proof, note that $A \to B_\beta$ satisfy an analogue of (i.) and (ii.), hence the functor $A \to \varprojlim_\beta B_\beta$ is bijective on objects and weakly fully faithful, hence an equivalence of simplicial categories. $\qquad \square$

The following proposition will be of use in our induction later on, and is a good illustration of one of the lemmas we just proved.

Proposition 3.34 (Cisinski, Hirschowitz-Simpson). *Let $\mathcal{M}$ be a model category and $[n]$ a finite ordinal. Then the functor $\mathsf{Fun}([n], \mathcal{M}) \to \mathsf{Fun}([n], L\mathcal{M})$ induces an equivalence of quasicategories*

$$L\,\mathsf{Fun}([n], \mathcal{M}) \cong \mathsf{Fun}([n], L\mathcal{M}).$$

Proof. Equip $\mathsf{Fun}([n], \mathcal{M})$ with projective model structure. It will be sufficient to prove that the induced functor $L\,\mathsf{Fun}([n], \mathcal{M})_{cf} \cong \mathsf{Fun}([n], L\mathcal{M}_{cf})$ is an equivalence.

The map $[n-1] \coprod_{[0]} [1] \to [n]$ (we include $[n-1]$ as a left interval in $[n]$) is inner anodyne (one can express it as a pushout-join and apply [18, Lemma 2.1.2.3]). This implies that for any quasicategory $\mathcal{X}$, the induced square

$$
\begin{array}{ccc}
\mathsf{Fun}([n], \mathcal{X}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([n-1], \mathcal{X}) & \longrightarrow & \mathcal{X}
\end{array}
$$

is a pullback in $\mathbf{Cat}_\infty$. In particular, it is true for $L\mathcal{M}_{cf}$.

Note that we have a similar pullback square of 1-categories of fibrant-cofibrant objects:

$$
\begin{array}{ccc}
\mathsf{Fun}([n], \mathcal{M})_{cf} & \longrightarrow & \mathsf{Fun}([1], \mathcal{M})_{cf} \\
\downarrow & & \downarrow \\
\mathsf{Fun}([n-1], \mathcal{M})_{cf} & \longrightarrow & \mathcal{M}_{cf}
\end{array}
$$

where, again, we equip all functor categories with projective model structure. This implies that for any $X \in \mathsf{Fun}([n], \mathcal{M})_{cf}$ and a simplicial resolution $Y_\bullet \in \mathsf{Fun}([n], \mathcal{M})_{cf}^{\Delta^{\mathrm{op}}}$, the following square is a pullback in $\mathbf{SSet}$:

$$
\begin{array}{ccc}
\mathsf{Fun}([n], \mathcal{M})_{cf}(X, Y_\bullet) & \longrightarrow & \mathsf{Fun}([1], \mathcal{M})_{cf}(X, Y_\bullet) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([n-1], \mathcal{M})_{cf}(X, Y_\bullet) & \longrightarrow & \mathcal{M}_{cf}(X, Y_\bullet),
\end{array}
$$

where we abuse the notation and identify $X, Y_\bullet$ and their images in the different categories. Note that $Y_\bullet$ remains a simplicial resolution even when restricted to $\mathsf{Fun}([n-1], \mathcal{M})$ and all other categories. All simplicial sets participating in the diagram above are fibrant, and moreover, the map

$$
\mathsf{Fun}([1], \mathcal{M})_{cf}(X, Y_\bullet) \to \mathcal{M}_{cf}(X, Y_\bullet) \tag{3.10}
$$

is a Kan fibration: denoting by ev the evaluation-at-zero functor $\mathsf{Fun}([1], \mathcal{M}) \to \mathcal{M}$ and by $ev_!$ its left adjoint, we see that the map (3.10) is given by precomposing with $ev_! ev X \to X$, which is a projective cofibration by inspection.

Recall [23, Lemma 6.1] that given any model category $\mathcal{N}$, there exists a span of weak equivalences of simplicial sets relating the mapping spaces $L^H \mathcal{N}(X, Y)$ of the Dwyer-Kan localisation and homotopy function complexes $\mathcal{N}(X, Y_\bullet)$. Note that the span of [23, Lemma 6.1] is functorial, both with respect to maps in $\mathcal{N}$ and along functors $\mathcal{N}_{cf} \to \mathcal{N}'$ that preserve weak equivalences.

Applying the above observation to establish the equivalence between $\mathsf{Fun}([n], \mathcal{M})_{cf}(X, Y_\bullet)$ and $L^H \mathsf{Fun}([n], \mathcal{M})_{cf}(X, Y_0)$, we see that the conditions (i.) and (ii.) of Lemma 3.32 are satisfied. Since every isomorphism in $\mathrm{Ho}\,\mathcal{M}_{cf}$ can be realised as a weak equivalence between

fibrant-cofibrant objects, it is easy to check that the functor $\mathsf{Fun}([1], \mathcal{M})_{cf} \to \mathcal{M}_{cf}$ is a Ho-isofibration as well. Thus Lemma 3.32 (and a passage through homotopy coherent nerve) implies that

$$
\begin{array}{ccc}
L\,\mathsf{Fun}([n], \mathcal{M})_{cf} & \longrightarrow & L\,\mathsf{Fun}([1], \mathcal{M})_{cf} \\
\downarrow & & \downarrow \\
L\,\mathsf{Fun}([n-1], \mathcal{M})_{cf} & \longrightarrow & L\mathcal{M}_{cf},
\end{array}
$$

is a pullback in $\mathbf{Cat}_\infty$. Everything will thus follow by induction from the case $n = 1$.

The infinity-functor $L\,\mathsf{Fun}([1], \mathcal{M})_{cf} \cong \mathsf{Fun}([1], L\mathcal{M}_{cf})$ is essentially surjective. This is true since for any cofibrant-fibrant x, y, the map

$$
\mathcal{M}_{cf}(x, y) \to \pi_0 L\mathcal{M}_{cf}(x, y) \cong \mathrm{Ho}\,\mathcal{M}_{cf}(x, y)
$$

is surjective, as per well known fact for model categories. It remains to prove fully faithfulness. Let $f : A \to B$ and $g_\bullet : C_\bullet \to D_\bullet$ be, respectively, an object of $L\,\mathsf{Fun}([1], \mathcal{M})_{cf}$ and a simplicial resolution in $L\,\mathsf{Fun}([1], \mathcal{M})_{cf}^{\Delta^{\mathsf{op}}}$. The following square is a pullback in $\mathbf{SSet}$:

$$
\begin{array}{ccc}
\mathsf{Fun}([1], \mathcal{M})_{cf}(f, g_\bullet) & \longrightarrow & \mathcal{M}_{cf}(A, C_\bullet) \\
\downarrow & & \downarrow \\
\mathcal{M}_{cf}(B, D_\bullet) & \longrightarrow & \mathcal{M}_{cf}(A, D_\bullet),
\end{array}
$$

and the bottom map is a fibration, being induced as a pullback along the cofibration $A \to B$. Using [23, Lemma 6.1], this implies again that the mapping space $L^H\,\mathsf{Fun}([1], \mathcal{M})_{cf}(f, g_0)$ is equivalent to the homotopy end of the functor

$$
[1]^{\mathsf{op}} \times [1] \xrightarrow{(f, g_0)} L^H \mathcal{M}_{cf}^{\mathsf{op}} \times L^H \mathcal{M}_{cf} \xrightarrow{\mathrm{Hom}} \mathbf{SSet},
$$

however, up to an application of the coherent nerve and a choice of a model of the hom-functor [19, Proposition 5.2.1.11], it is exactly the same presentation that is valid for the mapping spaces in $\mathsf{Fun}([1], L\mathcal{M}_{cf})$, [9, Proposition 5.1]. $\qquad\square$

3.3 The comparison

Recall that a left model Reedy fibration is a functor $p : \mathcal{M} \to \mathcal{R}$ to a Reedy category that is a Grothendieck opfibration, a Grothendieck fibration over $\mathcal{R}_-$, such that the associated semifibration (cf Lemma 1.41) is admissible in the sense of Definition 2.9.

Corollary 3.35. *The infinity-localisation along the fibrewise weak equivalences $Lp : L\mathcal{M} \to \mathcal{R}$ is universal, and is a left Reedy fibration of quasicategories. The functor $\mathcal{M} \to L\mathcal{M}$ preserves coCartesian arrows with cofibrant domain, and Cartesian arrows over $\mathcal{R}_-$ with fibrant codomain.*

Proof. An immediate specialisation of Corollary 3.29. $\qquad\square$

Remark 3.36 (Cf Remark 3.22). Using the calculus of zig-zags, it is possible to show that the saturated localiser giving rise to $L\mathcal{M}$ coincides in fact with the totality of weak equivalences: it suffices to see that a fibrewise map that gets inverted in $L\mathcal{M}$, is inverted in the fibre model category localisation $L\mathcal{M}(c)$. By choosing a zig-zag representative for the inverse, the statement becomes trivial, since a Reedy category has no non-identity isomorphisms.

The infinity-functor $\mathcal{M} \to L\mathcal{M}$ induces the functor $\mathsf{Sect}(\mathcal{R}, \mathcal{M}) \to \mathsf{Sect}(\mathcal{R}, L\mathcal{M})$ from (the nerve of) the 1-category $\mathsf{Sect}(\mathcal{R}, \mathcal{M})$ to the sections of the localisation $L\mathcal{M} \to \mathcal{R}$. Since fibrewise weak equivalences are inverted when projected to $L\mathcal{M}$, we get a well defined infinity-functor

$$L\mathsf{Sect}(\mathcal{R}, \mathcal{M}) \to \mathsf{Sect}(\mathcal{R}, L\mathcal{M})$$

from the quasicategorical localisation of the model structure on $\mathsf{Sect}(\mathcal{R}, \mathcal{M})$.

Theorem 3.37. *Let $\mathcal{M} \to \mathcal{R}$ be a left model Reedy fibration. Then the induced infinity-functor $L\mathsf{Sect}(\mathcal{R}, \mathcal{M}) \to \mathsf{Sect}(\mathcal{R}, L\mathcal{M})$ is a categorical equivalence.*

The proof will rely on the inductive description of sections over Reedy categories. For $\mathsf{Sect}(\mathcal{R}, L\mathcal{M})$, we have Proposition 3.17 and, evidently, the fact that for a good filtration $\mathcal{R}_\beta$, we have that $\mathsf{Sect}(\mathcal{R}, L\mathcal{M}) \cong \varprojlim \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})$, where the limit is taken in $\mathbf{Cat}_\infty$ along the filtration of $\mathcal{R}$. Let us see if the similar description holds for $L\mathsf{Sect}(\mathcal{R}, \mathcal{M})$.

Given any model category $\mathcal{X}$, denote by $\mathcal{X}_{c \to f} \subset \mathsf{Fun}([1], \mathcal{X})$ the full subcategory given by $X \to Y$ with X cofibrant and Y fibrant. Similarly, denote $\mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f} \subset \mathsf{Fun}([2], \mathcal{X})$ the full subcategory of $X \to Y \to Z$ with X cofibrant, Z fibrant, $X \to Y$ a cofibration, $Y \to Z$ a fibration. These categories appear naturally as targets for latching-matching decompositions.

Proposition 3.38. *Let $\mathcal{X}$ be a model category.*

1. *The inclusions $\mathcal{X}_{c \to f} \subset \mathsf{Fun}([1], \mathcal{X})$ and $\mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f} \subset \mathsf{Fun}([2], \mathcal{X})$ induce weak equivalences of infinity-localisations.*

2. *The functor $\mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f} \to \mathcal{X}_{c \to f}$ sending $X \to Y \to Z$ to $X \to Z$ induces an isofibration after taking (ordinary) localisations, $\mathrm{Ho}\, \mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f} \to \mathrm{Ho}\, \mathcal{X}_{c \to f}$.*

Proof.
1. We will do the case of $\mathcal{X}_{c \to f} \subset \mathsf{Fun}([1], \mathcal{X})$, with the second case being similar but more

cumbersome. Denote by $\mathcal{Y}$ the category which objects are diagrams

$$
\begin{array}{ccc}
X_1 & \longrightarrow & Y_1 \\
{\scriptstyle\sim}\uparrow & & \uparrow{\scriptstyle\sim} \\
QX_2 & \longrightarrow & Y_2 \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
QX_3 & \longrightarrow & RY_3
\end{array}
\tag{3.11}
$$

where the vertical maps are weak equivalences, the objects whose name starts with Q are cofibrant, and the objects whose name starts with R are fibrant. Various projections from $\mathcal{Y}$ define a diagram of weak equivalence preserving functors fitting together in a 2-diagram

$$
\begin{array}{c}
\mathcal{Y} \\
\swarrow \quad \downarrow \quad \searrow \\
\mathcal{X}_{c\to f} \xleftarrow{\ \Leftarrow\ } \mathcal{X}_{c\to *} \xrightarrow{\ \Rightarrow\ } \mathcal{X}_{*\to *}.
\end{array}
$$

Here, as the names suggest, $\mathcal{X}_{*\to *} = \mathsf{Fun}([1], \mathcal{X})$ and $\mathcal{X}_{c\to *}$ is its full subcategory of arrows with cofibrant domain. Both depicted natural transformations are valued in $\mathcal{W}$. Moreover, the inclusion $\mathcal{X}_{c\to f} \subset \mathcal{X}_{*\to *}$ can be factored as $\mathcal{X}_{c\to f} \to \mathcal{Y} \to \mathcal{X}_{*\to *}$. Here, the first functor is a homotopy inverse (in the sense of [4]) to the depicted $\mathcal{Y} \to \mathcal{X}_{c\to f}$, given by putting $QX \to RY$ in all three rows of (3.11).

In general, given two localisers $(\mathcal{M}, \mathcal{W}_\mathcal{M}), (\mathcal{N}, \mathcal{W}_\mathcal{N})$ one has the following diagram of (higher) categories

$$
\mathsf{Fun}'(\mathcal{M}, \mathcal{N}) \longrightarrow \mathsf{Fun}'(\mathcal{M}, L\mathcal{N}) \xleftarrow{\ \sim\ } \mathsf{Fun}(L\mathcal{M}, L\mathcal{N})
$$

with primes indicating (weak) equivalence preserving functors. As a consequence, a natural transformation $\alpha : F \Rightarrow G$ of weak equivalence preserving functors induces a natural transformation $\bar{\alpha} : \bar{F} \to \bar{G}$ of infinity-functors from $L\mathcal{M}$ to $L\mathcal{N}$; if α took values in $\mathcal{W}_\mathcal{N}$ then $\bar{\alpha}$ is invertible. Thus everything will follow if we prove that both functors

$$
\mathcal{X}_{c\to f} \leftarrow \mathcal{X}_{c\to *} \to \mathcal{X}_{*\to *}
$$

are weak equivalences in the model structure of [4]. Let us first consider the case of $\mathcal{X}_{c\to *} \to \mathcal{X}_{*\to *}$. We shall prove that it is a left approximation. Given $X_\bullet \to Y_\bullet$ in $\mathsf{Fun}([n], \mathcal{X}_{*\to *})$, we have to prove the contractibility of the category $\mathcal{C}(X_\bullet \to Y_\bullet)$ of diagrams

$$
\begin{array}{ccc}
QX_\bullet & \longrightarrow & Y'_\bullet \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
X_\bullet & \longrightarrow & Y_\bullet
\end{array}
$$

with bottom row fixed. There is a functor $\mathcal{C}(X_\bullet \to Y_\bullet) \to \mathcal{C}(X_\bullet)$, where $\mathcal{C}(X_\bullet)$ is the category of objectwise cofibrant replacements of $X_\bullet$, and hence it is contractible. The functor $\mathcal{C}(X_\bullet \to Y_\bullet) \to \mathcal{C}(X_\bullet)$ is furthermore a Grothendieck fibration, so it will suffice to show that its fibres have contractible nerve. The fibres are seen to have final objects, given by diagrams

$$
\begin{array}{ccc}
QX_\bullet & \longrightarrow & Y_\bullet \\
\sim \downarrow & & \downarrow = \\
X_\bullet & \longrightarrow & Y_\bullet .
\end{array}
$$

In the case of $\mathcal{X}_{c\to f} \leftarrow \mathcal{X}_{c\to *}$, we are, using the right approximation argument, presented with studying the category $\mathcal{D}(QX_\bullet \to Y_\bullet)$ of diagrams

$$
\begin{array}{ccc}
QX_\bullet & \longrightarrow & Y_\bullet \\
\sim \downarrow & & \downarrow \sim \\
QX'_\bullet & \longrightarrow & RY_\bullet
\end{array}
\qquad (3.12)
$$

with fixed first row, and we instead consider the projection $\mathcal{D}(QX_\bullet \to Y_\bullet) \to \mathcal{D}(Y_\bullet)$ to the category of fibrant replacements of $Y_\bullet$. The fibres of this projection are given by diagrams (3.12) with top row and left column fixed, and again,

$$
\begin{array}{ccc}
QX_\bullet & \longrightarrow & Y_\bullet \\
= \downarrow & & \downarrow \sim \\
QX_\bullet & \longrightarrow & RY_\bullet
\end{array}
$$

serves as an initial object.

2. Consider first the following situation:

$$
\begin{array}{ccc}
X & \hookrightarrow Y \twoheadrightarrow & Z \\
\sim \downarrow & & \downarrow \sim \\
X' & \longrightarrow & Z'
\end{array}
$$

with the top row in $\mathcal{X}_{c\hookrightarrow cf\twoheadrightarrow f}$ and the bottom row in $\mathcal{X}_{c\to f}$. Taking the pushout of the left corner, we factor the bottom map as follows:

$$
\begin{array}{ccccc}
X & \hookrightarrow Y & \twoheadrightarrow & Z \\
\sim \downarrow & \downarrow \sim & & \downarrow \sim \\
X' & \hookrightarrow Y'' & \longrightarrow & Z' .
\end{array}
$$

Since X, X', Y are cofibrant and $X \to Y$ is a cofibration, we see that the induced map $Y \to Y''$ is a weak equivalence. Factoring $Y'' \to Z'$ as a trivial cofibration $Y'' \hookrightarrow Y'$ followed

by a fibration $Y' \twoheadrightarrow Z'$, we get a modified factorisation of the map $X' \to Z'$, together with the diagram

$$
\begin{array}{ccc}
X & \hookrightarrow Y \longrightarrow\!\!\!\!\!\twoheadrightarrow & Z \\
\sim\downarrow & \quad\downarrow\sim & \quad\downarrow\sim \\
X' & \hookrightarrow Y' \longrightarrow\!\!\!\!\!\twoheadrightarrow & Z'
\end{array}
$$

in $\mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f}$.

Using a dual argument with pullbacks, one can treat the case of weak equivalences pointing up:

$$
\begin{array}{ccc}
X & \hookrightarrow Y \longrightarrow\!\!\!\!\!\twoheadrightarrow & Z \\
\sim\uparrow & & \quad\uparrow\sim \\
X' & \longrightarrow & Z'.
\end{array}
$$

In general, a weak equivalence in $\mathrm{Ho}\,\mathcal{X}_{c \to f}$ can be represented as a zig-zag of weak equivalences in $\mathcal{X}_{c \to f}$. Using the above arguments, the isofibration property of $\mathrm{Ho}\,\mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f} \to \mathrm{Ho}\,\mathcal{X}_{c \to f}$ is readily verified step by step. $\qquad\square$

Remark 3.39. The main difficulty of Proposition 3.38.1 comes from the non-assumption of the functoriality of factorisations. If the factorisations in $\mathcal{X}$ are functorial, it is easy to construct the (weakly) inverse functors to the inclusions $\mathcal{X}_{c \to f} \subset \mathsf{Fun}([1], \mathcal{X})$ and $\mathcal{X}_{c \hookrightarrow cf \twoheadrightarrow f} \subset \mathsf{Fun}([2], \mathcal{X})$.

Denote by $\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf}$ the full subcategory of cofibrant-fibrant sections for the Reedy model structure. Since $L\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf} \cong L\mathsf{Sect}(\mathcal{R}, \mathcal{M})$, Theorem 3.37 will follow if we prove that the induced functor $L\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf} \to \mathsf{Sect}(\mathcal{R}, L\mathcal{M})$ is a categorical equivalence.

Proposition 3.40. *Let $\mathcal{M} \to \mathcal{R}$ be a left model Reedy fibration. Choose a good filtration $\{\mathcal{R}_\beta\}_{\beta < \alpha}$. Then*

1. *if $\mathcal{R}_\beta$ is obtained from $\mathcal{R}_{<\beta}$ by adding $x \in \mathcal{R}$, then the assignments*

$$
X \mapsto (\mathscr{L}_x X \to X(x) \to \mathscr{M}_x X) \text{ and } X \mapsto (\mathscr{L}_x X \to \mathscr{M}_x X)
$$

   determine a pullback square in $\mathbf{Cat}_\infty$

$$
\begin{array}{ccc}
L\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} & \longrightarrow & L\,\mathsf{Fun}([2], \mathcal{M}(x)) \\
\downarrow & & \downarrow \\
L\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf} & \longrightarrow & L\,\mathsf{Fun}([1], \mathcal{M}(x)).
\end{array}
$$

2. *For each ordinal β, the $\mathbf{Cat}_\infty$-limit of $\{L\mathsf{Sect}(\mathcal{R}_\gamma, \mathcal{M})_{cf}\}_{\gamma < \beta}$ is equivalent, via the evident map, to $L\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf}$.*

3. *The* $\mathbf{Cat}_\infty$-*limit of* $\{L\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}\}_\beta$ *is equivalent, via the evident map, to* $L\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf}$.

Proof. Considering the relation between quasicategorical and Dwyer-Kan simplicial localisation as explained in [11], it will be sufficient to prove the corresponding statements for simplicial localisations. For this, we shall use Lemmas 3.32 and 3.33.

Using the same notations as before, there is a pullback square of categories

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} & \longrightarrow & \mathcal{M}(x)_{c \to cf \twoheadrightarrow f} \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf} & \longrightarrow & \mathcal{M}(x)_{c \to f}
\end{array}
\tag{3.13}
$$

with horizontal functors given by latching-matching factorisations. This square has the property that all functors preserve weak equivalences.

Take a cofibrant-fibrant section $X \in \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}$ and a(n always existing) simplicial resolution $\mathbf{Y} \in \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}^{\Delta^{\mathrm{op}}}$. Note that the restriction $X|_{\mathcal{R}_{<\beta}}$ remains cofibrant-fibrant and $\mathbf{Y}|_{\mathcal{R}_{<\beta}}$ remains a resolution valued in cofibrant-fibrant objects, since the condition for $\mathbf{Y}$ to be Reedy fibrant (as a simplicial object) is required objectwise. We thus have that the natural map

$$
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})(X, \mathbf{Y}) \to \mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})(X|_{\mathcal{R}_{<\beta}}, \mathbf{Y}|_{\mathcal{R}_{<\beta}})
$$

is a map of Kan complexes representing the corresponding map $L^H\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})(X, \mathbf{Y}(0)) \to L^H\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})(X|_{\mathcal{R}_{<\beta}}, \mathbf{Y}(0)|_{\mathcal{R}_{<\beta}})$ (we omit the subscript $(-)_{cf}$ as no information is lost).

Put the projective model structure on $\mathsf{Fun}([1], \mathcal{M}(x))$ (the unique map of $[1]$ raises the degree). For $\mathsf{Fun}([2], \mathcal{M}(x))$, consider the Reedy structure such that $0 \to 1$ and $0 \to 2$ are treated as degree-raising morphisms, and $1 \to 2$ as a degree-lowering. The vertical functor $\mathsf{Fun}([2], \mathcal{M}(x)) \to \mathsf{Fun}([1], \mathcal{M}(x))$ then preserves all classes of the model structure. The functor

$$
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} \longrightarrow \mathsf{Fun}([2], \mathcal{M}(x)), \quad X \mapsto X_{012} = \mathscr{L}_x X \to X(x) \to \mathscr{M}_x X,
$$

(we shall henceforth write $X_0 = \mathscr{L}_x X$ and so on) however, does not send fibrant-cofibrant objects to fibrant-cofibrant objects. Same observation applies to the functor

$$
\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf} \longrightarrow \mathsf{Fun}([1], \mathcal{M}(x)), \quad X \mapsto X_{02} = \mathscr{L}_x X \to \mathscr{M}_x X.
$$

Let us see how to correct this. For cofibrant-fibrant X, its value $X_{012} = X_0 \to X_1 \to X_2$ has the property that X_0 is cofibrant and $X_0 \to X_1$ is a cofibration. For our purposes, it will be sufficient to work with a cofibrant replacement QX_{012} that we obtain by factoring $X_0 \to X_2$ as a cofibration $X_0 \to QX_2$ followed by a trivial fibration $QX_2 \to X_2$. Since $X_0 \to X_1$ is a cofibration, the dotted arrow exists in the diagram

73

thus defining $QX_{012} \to X_{012}$. Note that the map resulting from composing these sequences, $QX_{02} \to X_{02}$, is a cofibrant replacement as well.

Similarly, taking a simplicial resolution $\mathbf{Y}$, we would like to arrange for its Reedy fibrant replacement. Since the transition functors of $\mathcal{M} \to \mathcal{R}$ along the "matching" maps $\mathcal{R}_-$ are right adjoints, and since the limits in $\mathsf{Sect}(-, \mathcal{M})$ are calculated fibre by fibre, we have the commutativity of $\mathscr{M}_x$ and the simplicial matching functor $\mathscr{M}_n^\Delta$. In particular, we have that the assignment

$$\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}^{\Delta^{\mathrm{op}}} \longrightarrow \mathsf{Fun}([2], \mathcal{M}(x))^{\Delta^{\mathrm{op}}}, \;\; \mathbf{Y} \mapsto \mathbf{Y}_{012} = \mathscr{L}_x \mathbf{Y} \to \mathbf{Y}(x) \to \mathscr{M}_x \mathbf{Y},$$

induces for each $[n] \in \Delta$ the diagram

$$
\begin{array}{ccccc}
\mathbf{Y}_0(n) & \longrightarrow & \mathbf{Y}_1(n) & \longrightarrow & \mathbf{Y}_2(n) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{M}_n^\Delta \mathbf{Y}_0 & \longrightarrow & \mathscr{M}_n^\Delta \mathbf{Y}_1 & \longrightarrow & \mathscr{M}_n^\Delta \mathbf{Y}_2
\end{array}
$$

such that the maps $\mathbf{Y}_2(n) \to \mathscr{M}_n^\Delta \mathbf{Y}_2$ and $\mathbf{Y}_1(n) \to \mathbf{Y}_2(n) \prod_{\mathscr{M}_n^\Delta \mathbf{Y}_2} \mathscr{M}_n^\Delta \mathbf{Y}_1$ (and hence $\mathbf{Y}_1(n) \to \mathscr{M}_n^\Delta \mathbf{Y}_1$) are fibrations. As a consequence, to get a fibrant replacement of $\mathbf{Y}_{012}$ it will suffice to factor $\mathbf{Y}_0 \to \mathbf{Y}_1$ as a trivial (simplicial) Reedy cofibration $\mathbf{Y}_0 \to R\mathbf{Y}_1$ followed by a Reedy fibration $R\mathbf{Y}_1 \to \mathbf{Y}_1$. Denote the result by $\mathbf{Y}_{012} \to R\mathbf{Y}_{012}$; the associated map $\mathbf{Y}_{02} \to R\mathbf{Y}_{02}$ is also a Reedy fibrant replacement in $\mathsf{Fun}([1], \mathcal{M}(x))^{\Delta^{\mathrm{op}}}$.

Observe now that, starting from $X \in \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}$ and $\mathbf{Y} \in \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}^{\Delta^{\mathrm{op}}}$ as above, we have the following pullback squares of simplicial sets:

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})(X, \mathbf{Y}) & \longrightarrow & \mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})(X|_{\mathcal{R}_{<\beta}}, \mathbf{Y}|_{\mathcal{R}_{<\beta}}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([2], \mathcal{M}(x))(X_{012}, \mathbf{Y}_{012}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{M}(x))(X_{02}, \mathbf{Y}_{02}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([2], \mathcal{M}(x))(QX_{012}, \mathbf{Y}_{012}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{M}(x))(QX_{02}, \mathbf{Y}_{02}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}([2], \mathcal{M}(x))(QX_{012}, R\mathbf{Y}_{012}) & \longrightarrow & \mathsf{Fun}([1], \mathcal{M}(x))(QX_{02}, R\mathbf{Y}_{02}).
\end{array}
$$

(3.14)

We thus have that the outer square is likewise a pullback. Moreover, observe that

$$\mathsf{Fun}([2], \mathcal{M}(x))(QX_{012}, R\mathbf{Y}_{012}) \longrightarrow \mathsf{Fun}([1], \mathcal{M}(x))(QX_{02}, R\mathbf{Y}_{02})$$

is a Kan fibration: it is induced by applying $\mathsf{Fun}([2], \mathcal{M}(x))(-, R\mathbf{Y}_{012})$ to the map

$$
\begin{array}{ccccc}
X_0 & \longrightarrow & X_0 & \longrightarrow & QX_2 \\
\downarrow & & \downarrow & & \downarrow \\
X_0 & \longrightarrow & X_1 & \longrightarrow & QX_2
\end{array}
$$

which is a cofibration in the chosen model structure on $\mathsf{Fun}([2], \mathcal{M}(x))$. This implies that

$$\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})(X, \mathbf{Y}) \longrightarrow \mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})(X|_{\mathcal{R}_{<\beta}}, \mathbf{Y}|_{\mathcal{R}_{<\beta}}) \qquad (3.15)$$

is a Kan fibration and that the outer square of (3.14) is a homotopy pullback.

Let us examine the induced pull-back square of simplicial localisations,

$$
\begin{array}{ccc}
L^H\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} & \longrightarrow & L^H\mathcal{M}(x)_{c \hookrightarrow cf \twoheadrightarrow f} \\
\downarrow & & \downarrow \\
L^H\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf} & \longrightarrow & L^H\mathcal{M}(x)_{c \to f}.
\end{array}
\qquad (3.16)
$$

The right vertical functor is a Ho-isofibration by (2.) of Proposition 3.38, and for each $X, Y \in L^H\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}$, the induced square of simplicial hom-spaces is a homotopy pullback, as follows from the argument above and [23, Lemma 6.1]. Lemma 3.32 thus tells us that (3.16) is a homotopy pullback. Combining with the equivalences

$$L^H\mathcal{M}(x)_{c \hookrightarrow cf \twoheadrightarrow f} \xrightarrow{\sim} L^H\mathsf{Fun}([2], \mathcal{M}(x)) \text{ and } L^H\mathcal{M}(x)_{c \to f} \xrightarrow{\sim} L^H\mathsf{Fun}([1], \mathcal{M}(x)),$$

we prove the first statement of the proposition.

The 1-category $\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf}$ is equivalent (even isomorphic) to the limit of the inverse system $\{\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf}\}_{\beta < \alpha}$. Let us verify that the conditions of Lemma 3.33 are satisfied for

$$L^H\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf} \to L^H\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} \to L^H\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf}.$$

The conditions (0.), (2.) and (i.) of Lemma 3.33 are readily verified.

Using the pullback diagram (3.13) and (2.) of Proposition 3.38, we conclude that for each β, the functor $L^H\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} \to L^H\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf}$ is a Ho-isofibration. This corresponds to the condition (1.) of Lemma 3.33. The limit ordinals and $\mathcal{R}$ itself are treated by the same sort of argument. If β is an ordinal, then $\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf}(X, \mathbf{Y})$ coincides with the limit of $\{\mathsf{Sect}(\mathcal{R}_\gamma, \mathcal{M})_{cf}(X|_{\mathcal{R}_\gamma}, \mathbf{Y}|_{\mathcal{R}_\gamma})\}_{\gamma < \beta}$. Using the Kan fibration remark around (3.15) and induction, we observe that $\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf}(X, \mathbf{Y})$ is also the homotopy limit of its restrictions. Inductively we also get that $\mathsf{Sect}(\mathcal{R}, \mathcal{M})_{cf}(X, \mathbf{Y})$ is the homotopy limit of its restrictions, and using [23, Lemma 6.1] again, we verify (3.) and (ii.). $\qquad \square$

Proof of Theorem 3.37. In light of Propositions 3.17 and 3.40, everything will follow by transfinite induction from the comparison for model categories as described in Proposition 3.34, if we show that for each β, there is a map in $\mathbf{Cat}_\infty$, naturally induced on each term by $F : \mathcal{M} \to L\mathcal{M}$, from the square

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} & \longrightarrow & \mathsf{Fun}([2], \mathcal{M}(x)) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf} & \longrightarrow & \mathsf{Fun}([1], \mathcal{M}(x))
\end{array}
\qquad (3.17)
$$

to the square

$$
\begin{array}{ccc}
\mathsf{Sect}(\mathcal{R}_\beta, L\mathcal{M}) & \longrightarrow & \mathsf{Fun}([2], L\mathcal{M}(x)) \\
\downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{R}_{<\beta}, L\mathcal{M}) & \longrightarrow & \mathsf{Fun}([1], L\mathcal{M}(x)).
\end{array}
\tag{3.18}
$$

For this, we observe that cofibrant-fibrant sections are (F, x)-compatible in the sense of Definition 3.18. Indeed, the functor $\mathcal{M} \to L\mathcal{M}$ preserves coCartesian arrows with cofibrant domain and Cartesian arrows over $\mathcal{R}_-$ with fibrant codomain; given a cofibrant-fibrant section S, its right restriction $R_x S : Mat(x) \to \mathcal{M}(x)$ is an injectively fibrant diagram, and hence its limit remains [2, Proposition 2.5.6] a limit in $L\mathcal{M}(x)$. A dual observation is true for the left restrictions. Therefore, the square (*3.17*) factors as

$$
\begin{array}{ccccc}
\mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{cf} & \longrightarrow & \mathsf{Sect}(\mathcal{R}_\beta, \mathcal{M})_{(F,x)} & \longrightarrow & \mathsf{Fun}([2], \mathcal{M}(x)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{cf} & \longrightarrow & \mathsf{Sect}(\mathcal{R}_{<\beta}, \mathcal{M})_{(F,x)} & \longrightarrow & \mathsf{Fun}([1], \mathcal{M}(x)).
\end{array}
$$

Using Proposition 3.20, we get the desired map. $\qquad\square$

We finish this part by recording the following essential statement, which already appears as [2, Theorem 2.5.9] and in [6].

Corollary 3.41. *Let $\mathcal{M}$ be a model category. Then the infinity-category $L\mathcal{M}$ is bicomplete.*

Proof. By duality, it is enough to prove the colimit part. Theorem 3.37 establishes an equivalence $L\,\mathsf{Fun}(\mathcal{R}, \mathcal{M}) \cong \mathsf{Fun}(\mathcal{R}, L\mathcal{M})$ for any Reedy category $\mathcal{R}$. Moreover, if $\mathcal{R}$ is direct, then we have a Quillen adjunction

$$
\varinjlim_{\mathcal{R}} : \mathsf{Fun}(\mathcal{R}, \mathcal{M}) \rightleftarrows \mathcal{M} : c_{\mathcal{R}}
$$

that by Corollary 3.31 gives an infinity-adjunction

$$
\mathbb{L}\varinjlim_{\mathcal{R}} : \mathsf{Fun}(\mathcal{R}, L\mathcal{M}) \rightleftarrows L\mathcal{M} : \mathbb{R}c_{\mathcal{R}}.
$$

We thus conclude that $L\mathcal{M}$ has colimits of shape $\mathcal{R}$ for each direct Reedy category $\mathcal{R}$. But in particular that means the existence of pushouts and arbitrary coproducts, and by [18, Proposition 4.4.2.6] $L\mathcal{M}$ thus has all small colimits. $\qquad\square$

3.4 On Quillen presheaves

We conclude this paper by applying Theorem 3.37 to the presheaves of model categories and Quillen adjunctions.

It is known [6, Theorem 7.9.8] that for a model category $\mathcal{M}$ and a small category $\mathcal{C}$, one has the equivalence $L\,\mathsf{Fun}(\mathcal{C}, \mathcal{M}) \to \mathsf{Fun}(\mathcal{C}, L\mathcal{M})$, even if $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ has no obvious model structure if $\mathcal{M}$ is not cofibrantly generated. The theory of Quillen presheaves is rich in structure, so it is perhaps of no surprise that the same observation applies in this setting:

76

Proposition 3.42. *(Cf [16, Conjecture 18.3]) Let $\mathcal{M} \to \mathcal{C}$ be a Quillen presheaf over a small category $\mathcal{C}$. Then localising $\mathcal{M}$ along the fibrewise weak equivalences yields an equivalence of infinity-categories*

$$L\mathsf{Sect}(\mathcal{C}, \mathcal{M}) \xrightarrow{\sim} \mathsf{Sect}(\mathcal{C}, L\mathcal{M}).$$

In other words, sections of Quillen presheaves can be strictified over an arbitrary 1-categorical base.

It will be convenient to prove this proposition by the end of this subsection.

Remark 3.43. The proof of Proposition 3.42 and of all subsequent results depends only on one adjoint. Thus everything can be generalised to the case of an opfibration in model categories $\mathcal{M} \to \mathcal{C}$ which transition functors preserve cofibrations, trivial cofibrations, and colimits. We have chosen to work with Quillen presheaves as almost all the examples of colimit-preserving functors between model categories admit right adjoints.

Given a Quillen presheaf $\mathcal{M} \to \mathcal{C}$, consider a map $f : c \to c'$ and the induced adjunction $f_! : \mathcal{M}(c) \rightleftarrows \mathcal{M}(c') : f^*$. Corollary 3.31 supplies us with the infinity-adjunction $\mathbb{L}f_! : L\mathcal{M}(c) \rightleftarrows L\mathcal{M}(c') : \mathbb{R}f^*$, and we shall use the same notation to denote the induced adjunction on the homotopy categories: $\mathbb{L}f_! : \mathrm{Ho}\,\mathcal{M}(c) \rightleftarrows \mathrm{Ho}\,\mathcal{M}(c') : \mathbb{R}f^*$.

Lemma 3.44. *Let $\mathcal{M} \to \mathcal{C}$ be a Quillen presheaf, and S_l, S_r two sets of arrows of $\mathcal{C}$. Then the equivalence of Proposition 3.42 induces an equivalence*

$$L\mathsf{Sect}_{(S_l, S_r)}(\mathcal{C}, \mathcal{M}) \xrightarrow{\sim} \mathsf{Sect}_{(S_l, S_r)}(\mathcal{C}, L\mathcal{M})$$

to the infinity category $\mathsf{Sect}_{(S_l, S_r)}(\mathcal{C}, L\mathcal{M})$ of S_l-coCartesian and S_r-Cartesian sections, from the localisation of the full subcategory $\mathsf{Sect}_{(S_l, S_r)}(\mathcal{C}, \mathcal{M}) \subset \mathsf{Sect}(\mathcal{C}, \mathcal{M})$ consisting of all sections $S : \mathcal{C} \to \mathcal{M}$ such that

    i. For each $f : c \to c' \in S_l$, the induced map $\mathbb{L}f_! S(c) \to S(c')$ is an isomorphism in $\mathrm{Ho}\,\mathcal{M}(c')$,

    ii. For each $f : c \to c' \in S_r$, the induced map $S(c) \to \mathbb{R}f^* S(c')$ is an isomorphism in $\mathrm{Ho}\,\mathcal{M}(c)$.

Proof. Follows directly from the observations of Corollaries 3.29 and 3.31. $\qquad\square$

Corollary 3.45. *Let $\mathcal{M} \to \mathcal{C}$ be a Quillen presheaf. Then the higher-categorical limit of the covariant (respectively contravariant) diagram $x \mapsto L\mathcal{M}(x)$ is given by the infinity-localisation $L\mathsf{Sect}_{(\mathcal{C}, \emptyset)}(\mathcal{C}, \mathcal{M})$, (respectively $L\mathsf{Sect}_{(\emptyset, \mathcal{C})}(\mathcal{C}, \mathcal{M})$) consisting of those sections S such that, given any $f : c \to c'$, the induced map $\mathbb{L}f_! S(c) \to S(c')$ (respectively $S(c) \to \mathbb{R}f^* S(c')$) is an isomorphism in the homotopy category.*

Proof. Direct consequence of [18, Corollary 3.3.3.2], which is itself a consequence of the naturality of the Grothendieck construction [20, Remark 3.1.13]. $\qquad\square$

Quillen presheaves are closely related to the notion of descent. For an exemplary statement, let $B^\bullet : \Delta \to \mathcal{C}$ be a cosimplicial diagram in $\mathcal{C}$, and $A \to B^\bullet$ a natural transformation from the (constant diagram given by) $A \in \mathcal{C}$. For a Quillen presheaf $\mathcal{M} \to \mathcal{C}$, denote by $\mathsf{Sect}(B^\bullet, \mathcal{M}) = \mathsf{Sect}(\Delta, (B^\bullet)^*\mathcal{M})$.

Definition 3.46. Let $\mathcal{M} \to \mathcal{C}$ be a Quillen presheaf. A morphism $A \to B^\bullet$ in the notation above satisfies the descent property with respect to $\mathcal{M} \to \mathcal{C}$ if the induced functor

$$\mathcal{M}(A) \to \mathsf{Sect}_{\mathbb{L}cart}(B^\bullet, \mathcal{M})$$

is a weak equivalence of relative categories, where $\mathsf{Sect}_{\mathbb{L}cart}(B^\bullet, \mathcal{M})$ is the sub-category of sections $X : \Delta \to \mathcal{M}$ such that for each $\alpha : [n] \to [m]$, the induced map $\mathbb{L}\alpha_! X(n) \to X(m)$ is an equivalence.

Corollary 3.47. *In the situation above, if $A \to B^\bullet$ satisfies the descent property with respect to $\mathcal{M} \to \mathcal{C}$, then the map $A \to B^\bullet$ exhibits $L\mathcal{M}(A)$ as a $\mathbf{Cat}_\infty$-limit of $L\mathcal{M}(B^\bullet)$.*

Proof. Immediate. $\square$

Remark 3.48. Observe that the functor $\mathcal{M}(A) \to \mathsf{Sect}_{\mathbb{L}cart}(B^\bullet, \mathcal{M})$ factors as

$$\mathcal{M}(A) \to \mathsf{Fun}(\Delta, \mathcal{M}(A)) \to \mathsf{Sect}_{\mathbb{L}cart}(B^\bullet, \mathcal{M})$$

where the first functor is the constant diagram inclusion, and the second one is induced by applying $\mathsf{Sect}(-, \mathcal{M})$ to the map of the cosimplicial diagrams $f : A \to B^\bullet$. By Theorem 3.37 and Corollary 3.41, there is an adjunction

$$\mathbb{L}const : L\mathcal{M}(A) \rightleftarrows \mathsf{Fun}(\Delta, L\mathcal{M}(A)) \cong L\,\mathsf{Fun}(\Delta, \mathcal{M}(A)) : \mathbb{R}\varprojlim,$$

even though $\mathbb{R}\varprojlim$ does not in general come from a Quillen functor. We also have an induced infinity-adjunction

$$\mathbb{L}f_! : L\,\mathsf{Fun}(\Delta, \mathcal{M}(A)) \rightleftarrows L\mathsf{Sect}(B^\bullet, \mathcal{M}) : \mathbb{R}f^*$$

where $f_! \dashv f^*$ is the Quillen adjunction induced by $f : A \to B^\bullet$. Thus to check that $L\mathcal{M}(A) \to L\mathsf{Sect}_{\mathbb{L}cart}(B^\bullet, \mathcal{M})$ is an equivalence, it is enough to check that the infinity-adjunction $\mathbb{L}f_!\mathbb{L}const \dashv \mathbb{R}\varprojlim\mathbb{R}f^*$ is an adjoint equivalence. This, however, can be done on the level of homotopy categories. Thus the already known theory of descent for Quillen presheaves (see e.g. [30, Lemma 2.2.2.13]) generalises to our setting.

There are many examples of Quillen presheaves that live over a base which is equipped with a subcategory of weak equivalences. Following [15], we introduce

Definition 3.49. A *relative Quillen presheaf* over a category with weak equivalences $(\mathcal{C}, \mathcal{W}_\mathcal{C})$ is a Quillen presheaf $\mathcal{M} \to \mathcal{C}$ such that for each $f : c \to c'$ in $\mathcal{W}_\mathcal{C}$ the induced Quillen adjunction $f_! : \mathcal{M}(c) \rightleftarrows \mathcal{M}(c') : f^*$ is a Quillen equivalence.

Note that if we take the associated "straightened" functor $St(\mathcal{M}) : c \mapsto \mathcal{M}(c)$ and postcompose with the localisation, the induced infinity-functor $LSt(\mathcal{M}) : \mathcal{C} \to \mathbf{Cat}_\infty$ sends the maps of $\mathcal{W}_\mathcal{C}$ to equivalences of infinity-categories. It thus comes from an essentially unique functor $\overline{LSt(\mathcal{M})} : L\mathcal{C} \to \mathbf{Cat}_\infty$ by applying pullback along $\mathcal{C} \to L\mathcal{C}$.

If $(\mathcal{C}, \mathcal{W}_\mathcal{C})$ is small, we can consider the category $\mathsf{Sect}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, \mathcal{M})$ consisting of those sections S such that for each $f : c \to c'$ in $\mathcal{W}_\mathcal{C}$, both $\mathbb{L}f_! S(c) \to S(c')$ and $S(c) \to \mathbb{R}f^* S(c')$ are isomorphisms in the homotopy category.

Lemma 3.50. *Given a relative category $(\mathcal{C}, \mathcal{W}_\mathcal{C})$ and a categorical fibration $\mathcal{X} \to L\mathcal{C}$, the pull-back operation induces a categorical equivalence* $\mathsf{Sect}(L\mathcal{C}, \mathcal{X}) \cong \mathsf{Sect}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, \mathcal{X})$*, with the latter denoting the subcategory of sections $\mathcal{C} \to \mathcal{X}$ which send $\mathcal{W}_\mathcal{C}$ to equivalences of $\mathcal{X}$.*

Proof. We have a $(\mathbf{Cat}_\infty$-pullback$)$ diagram

$$
\begin{array}{ccc}
\mathsf{Fun}(L\mathcal{C}, \mathcal{X}) & \xrightarrow{\sim} & \mathsf{Fun}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}(L\mathcal{C}, L\mathcal{C}) & \xrightarrow{\sim} & \mathsf{Fun}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, L\mathcal{C})
\end{array}
\tag{3.19}
$$

with $\mathsf{Fun}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, \mathcal{X})$ (and similarly $\mathsf{Fun}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, L\mathcal{C})$) denoting the infinity-category of functors $\mathcal{C} \to \mathcal{X}$ that send $\mathcal{W}_\mathcal{C}$ to equivalences of $\mathcal{X}$. Both vertical maps in (*3.19*) are categorical fibrations, so taking strict $\mathbf{SSet}$-pullbacks over $id_\mathcal{C}$ and $\mathcal{C} \to L\mathcal{C}$ induces the sought-after equivalence $\mathsf{Sect}(L\mathcal{C}, \mathcal{X}) \cong \mathsf{Sect}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, \mathcal{X})$.

Proposition 3.51. *Let $\mathcal{M} \to \mathcal{C}$ be a Quillen presheaf over a small localiser $(\mathcal{C}, \mathcal{W}_\mathcal{C})$. Then the infinity-category $L\mathsf{Sect}_{\mathcal{W}_\mathcal{C}}(\mathcal{C}, \mathcal{M})$ is naturally equivalent to the sections $\mathsf{Sect}(L\mathcal{C}, \int \overline{LSt(\mathcal{M})})$ of the unstraightening $\int \overline{LSt(\mathcal{M})}) \to L\mathcal{C}$.*

Proof. Localising $\mathcal{M}$ along the fibrewise weak equivalences, we see that there is a following diagram of $\mathbf{Cat}_\infty$-pullbacks:

$$
\begin{array}{ccccc}
L\mathcal{M} & \xrightarrow{\sim} & \int LSt(\mathcal{M}) & \longrightarrow & \int \overline{LSt(\mathcal{M})}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{=} & \mathcal{C} & \longrightarrow & L\mathcal{C}.
\end{array}
\tag{3.20}
$$

The second square is a pullback due to the naturality of the Grothendieck construction. The first one is the adjoint of the map of $\mathbf{Cat}_\infty$-functors $StL(\mathcal{M}) \to LSt(\mathcal{M})$; the latter being an equivalence is a reformulation of the universality of the localisation $L\mathcal{M} \to \mathcal{C}$.

Combining Lemma 3.50 with (*3.20*), we see that $\mathsf{Sect}(L\mathcal{C}, \int \overline{LSt(\mathcal{M})})$ is canonically identified with the subcategory of those S in $\mathsf{Sect}(\mathcal{C}, L\mathcal{M})$ that send $\mathcal{W}_\mathcal{C}$ to Cartesian (or, equivalently, coCartesian) maps of $L\mathcal{M}$. The conclusion is then reached using Lemma 3.44. $\qquad\square$

Remark 3.52. The above proposition shows that $L\mathsf{Sect}_{\mathcal{W}_{\mathcal{C}}}(\mathcal{C}, \mathcal{M})$ serves as a strict model for the lax limit of the $\mathbf{Cat}_\infty$-diagram naturally associated to the data of a relative Quillen presheaf. The situation with the lax colimit is more intricate. In effect, one would like to localise $\mathcal{M}$ along the following class of weak equivalences $\mathcal{W}(\mathcal{M})$: those maps $\alpha : X \to Y$ such that the corresponding $\mathcal{C}$-map $f : x \to y$ belongs to $\mathcal{W}$, and either $\mathbb{L}f_! X \to Y$ or $X \to \mathbb{R}f^* Y$ is an equivalence. However, even with $\mathcal{W}_{\mathcal{C}}$ saturated, we cannot a priori guarantee that the described class $\mathcal{W}(\mathcal{M})$ is saturated as well. Thus one cannot conclude that there is an equivalence between $L_{\mathcal{W}(\mathcal{M})}\mathcal{M}$ and $\overline{LSt(\mathcal{M})}$.

The work [15] proves that when $\mathcal{C}$ is a model category, the localiser $\mathcal{W}(\mathcal{M})$ is saturated. More generally, it seems that whenever the base admits a certain calculus of fractions, one can verify the saturation property of $\mathcal{W}(\mathcal{M})$. Conjecturally, $\mathcal{W}(\mathcal{M})$ should be saturated if $\mathcal{W}_{\mathcal{C}}$ is.

Let us finally prove Proposition 3.42. Our strategy will be to replace any small category by a suitable Reedy category. Given a category $\mathcal{C}$, denote by $\Delta\mathcal{C}$ its category of simplices: the objects of $\Delta\mathcal{C}$ are given by functors $\sigma : [n] \to \mathcal{C}$, and the morphisms are natural transformations $[n] \to [n']$ compatible with maps to $\mathcal{C}$. The category $\Delta\mathcal{C}$ is a Reedy category with fibrant constants [8, 22.10], and the assignment $\sigma \mapsto \sigma(n)$ defines a functor $p : \Delta\mathcal{C} \to \mathcal{C}$, called p_t in [8, 22.11]. The comma-fibres of this functor are moreover related to the comma-fibres of the original category: one has $p/c = \Delta(\mathcal{C}/c)$. We shall henceforth write $\Delta\mathcal{C}/c$ to denote both categories. The inclusion functor $\Delta\mathcal{C}/c \to \Delta\mathcal{C}$ is seen to identify the latching categories: $Lat(\sigma, \sigma(n) \to c)$ computed in $\Delta\mathcal{C}/c$, is isomorphic to $Lat(\sigma)$ computed in $\Delta\mathcal{C}$, with the map to c being automatically supplied. This fact will be useful later on.

Our first step is the proof that $p : \Delta\mathcal{C} \to \mathcal{C}$ presents $\mathcal{C}$ as the higher-categorical localisation of $\Delta\mathcal{C}$ along the subset $\mathcal{W}$ consisting of those maps that are sent to identities by p. The proof that we give below is the corrected version [31, Proposition A.1]: there, the authors in particular state that p is a coCartesian fibration, and such a claim is false (unfortunately, we committed the same mistake in an earlier version of this paper). For a different write-up, we invite the reader to consult [29, 5.3].

Lemma 3.53. *The functor $p : \Delta\mathcal{C} \to \mathcal{C}$ is an infinity-localisation along the p-identities $\mathcal{W}$, meaning that for any infinity-category $\mathcal{X}$, the infinity-functor $p^* : \mathsf{Fun}(\mathcal{C}, \mathcal{X}) \to \mathsf{Fun}(\Delta\mathcal{C}, \mathcal{X})$ is full and faithful and its essential image consists of all those functors $F : \Delta\mathcal{C} \to \mathcal{X}$ that send $\mathcal{W}$ to equivalences in $\mathcal{X}$.*

Proof. Factoring p as $\Delta\mathcal{C} \xrightarrow{l} L_{\mathcal{W}}\Delta\mathcal{C} \xrightarrow{\pi} \mathcal{C}$ with $L_{\mathcal{W}}\Delta\mathcal{C}$ being the infinity-localisation, it remains to prove that π is an equivalence of (infinity)-categories. Applying the homotopy category functor to this diagram would give the usual factorisation $\Delta\mathcal{C} \to \mathrm{Ho}\,\Delta\mathcal{C} \to \mathcal{C}$ through the one-categorical localisation of $\Delta\mathcal{C}$; this observation permits to conclude that π is essentially surjective.

To show that π is fully faithful, one can use Yoneda lemma which tells us that it is enough [6, Proposition 4.5.2 (iii)] to verify that the induced left Kan extension $\pi_! : \mathsf{Fun}(L_{\mathcal{W}}\Delta\mathcal{C}, \mathcal{S}) \to$

$\mathsf{Fun}(\mathcal{C}, \mathcal{S})$ is fully faithful, with $\mathcal{S}$ denoting the infinity-category of spaces. We will show that $\pi_!$ is part of an adjoint equivalence.

We observe the existence of the following diagram,

$$
\begin{array}{ccc}
& \mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{S}) & \\
{\scriptstyle l^*} \nearrow & & \searrow {\scriptstyle p^*} \\
\mathsf{Fun}(L_{\mathcal{W}}\Delta\mathcal{C}, \mathcal{S}) \xleftarrow[\pi^*]{} & & \mathsf{Fun}(\mathcal{C}, \mathcal{S}),
\end{array}
$$

with $\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{S})$ meaning those infinity-functors that send $\mathcal{W}$ to equivalences. The functor l^* is an equivalence by definition. The functor π^* is right adjoint to $\pi_!$; it will thus suffice to show that p^* is an equivalence.

The left Kan extension along p restricts to $\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{S})$, giving an infinity-adjunction

$$
p_! : \mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{S}) \rightleftarrows \mathsf{Fun}(\mathcal{C}, \mathcal{S}) : p^*.
$$

The surjectivity of p on objects implies conservativity of p^*. Thus if we prove that the unit map $id \to p^* p_!$ is an equivalence, the rest will follow from the triangle identities ([6, Theorem 6.1.23]; alternatively one can pass to the homotopy adjunction and use the associated triangle identities). The pointwise expression for Kan extensions, [6, Proposition 6.4.9], implies that for each $X \in \mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{S})$, the unit map evaluated at $\tau : [n] \to \mathcal{C}$ with $\tau(n) = c$ takes the form

$$
X(\tau) \to \varinjlim_{\Delta\mathcal{C}/c} X|_{\Delta\mathcal{C}/c}. \tag{3.21}
$$

As per [8, 23.5], denote by $p^{-1}c$ the subcategory of $\Delta\mathcal{C}$ consisting of those simplices $\sigma : [m] \to \mathcal{C}$ such that $\sigma(m) = c$ and those maps $[m] \to [m']$ that map m to m'. The notation is from [8] and is abusive: the category $p^{-1}c$ is **not** the fibre of p at c. There is however an obvious functor $i : p^{-1}c \to \Delta\mathcal{C}/c$ sending σ to $(\sigma, id : \sigma(n) = c)$ and this functor admits a left adjoint, sending $(\sigma : [n] \to \mathcal{C}, f : \sigma(n) \to c)$ to the concatenated simplex $\sigma * f : [n+1] \to \mathcal{C}$ that is σ on first $n+1$ elements and f on the remaining edge (if we took the naive fibre we would not have an adjunction). The functor i is thus cofinal. Considering then the composition

$$
[0] \xrightarrow{\{\tau\}} p^{-1}c \xrightarrow{i} \Delta\mathcal{C}/c
$$

and taking the colimit of $X|_{\Delta\mathcal{C}/c}$ and its pullbacks at each category, we factor the map (*3.21*) as

$$
X(\tau) \to \varinjlim_{p^{-1}c} X|_{p^{-1}c} \xrightarrow{\cong} \varinjlim_{\Delta\mathcal{C}/c} X|_{\Delta\mathcal{C}/c}.
$$

Since X belongs to $\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{S})$, its restriction to $p^{-1}c$ sends all maps to equivalences in $\mathcal{S}$. The remaining fact that $X(\tau) \to \varinjlim_{p^{-1}c} X|_{p^{-1}c}$ is an equivalence then follows from Lemma 3.54 below since the category $p^{-1}c$ is contractible, possessing the initial object given by the zero-simplex c. $\qquad\square$

Lemma 3.54. *Let $\mathfrak{X}$ be a cocomplete quasicategory, K a contractible simplicial set and $x \in K$ a vertex. Then for any $F : K \to \mathfrak{X}$ sending all edges to equivalences, the natural map $F(x) \to \varinjlim_K F$ is an equivalence.*

Proof. In $\mathbf{SSet}_+$ factor $K^\flat \to \Delta^0$ as a trivial cofibration $K^\flat \to LK^\natural$ followed by a fibration $LK^\natural \to \Delta^0$. The quasicategory LK is the localisation of K with respect to all edges, hence LK is a Kan complex. Moreover the functor F induces an infinity-functor $\bar{F} : LK \to \mathfrak{X}$. Both the map $K \to LK$ and $x : [0] \to K \to LK$ are moreover cofinal, as per [18, Corollary 4.1.2.6]. We thus have $\varinjlim_K F \cong \varinjlim_{LK} \bar{F}$ and $\varinjlim_{LK} \bar{F} \cong \bar{F}(x) \cong F(x)$. $\qquad\square$

Proof of Proposition 3.42. By Lemma 3.53, the functor $p : \Delta\mathcal{C} \to \mathcal{C}$ is the localisation along the maps $\mathcal{W}$ that are sent to identities by p (to distinguish, the fibrewise weak equivalences over $c \in \mathcal{C}$ will be denoted $\mathcal{W}(c)$). We can assume the validity of Proposition 3.42 and its corollaries for Quillen presheaves over Reedy categories. Given a general Quillen presheaf $\mathcal{M} \to \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
p^*L\mathcal{M} & \longrightarrow & L\mathcal{M} \\
\downarrow & & \downarrow \\
\Delta\mathcal{C} & \longrightarrow & \mathcal{C},
\end{array}
$$

together with Lemmas 3.50 and 3.44 imply the canonical identification of $\mathsf{Sect}(\mathcal{C}, L\mathcal{M})$ with the infinity-localisation of the category $\mathsf{Sect}_\mathcal{W}(\Delta\mathcal{C}, \mathcal{M})$ consisting of those sections $S : \Delta\mathcal{C} \to \mathcal{M}$ that send $\mathcal{W}$ to (fibrewise) equivalences of $\mathcal{M}$. Note that taking pull-backs of sections induces a functor

$$
p^* : \mathsf{Sect}(\mathcal{C}, \mathcal{M}) \to \mathsf{Sect}_\mathcal{W}(\Delta\mathcal{C}, \mathcal{M})
$$

that preserves pointwise weak equivalences. It remains to show that p^* is a weak equivalence of relative categories.

Observe that p^* possesses a left adjoint, given by

$$
p_!S(c) = \varinjlim_{\Delta\mathcal{C}/c} Res_c S|_{\Delta\mathcal{C}/c}
$$

where $Res_c : \mathcal{M}|_{\Delta\mathcal{C}/c} \to \mathcal{M}(c)$ is the usual covariant restriction to the fibre. Since $\Delta\mathcal{C}/c \subset \Delta\mathcal{C}$ identifies the latching categories, if S is a Reedy-cofibrant section, then $Res_c S|_{\Delta\mathcal{C}/c} : \Delta\mathcal{C}/c \to \mathcal{M}(c)$ can be checked to be a Reedy-cofibrant section as well (the verification is the same as in the proof of (1.) of Lemma 2.18). Using [8, 22.10] we see that the functor $p_!$ preserves weak equivalences between Reedy-cofibrant sections. Since p^* preserves weak equivalences, Corollary 3.31 implies the derived adjunction $\mathbb{L}p_! \dashv \mathbb{R}p^*$ between the localisations. It remains to check that the restriction

$$
\mathbb{L}p_! : L\mathsf{Sect}_\mathcal{W}(\Delta\mathcal{C}, \mathcal{M}) \rightleftarrows L\mathsf{Sect}(\mathcal{C}, \mathcal{M}) : \mathbb{R}p^*
$$

is an adjoint equivalence. For this, it is enough to verify that the corresponding homotopy category adjunction $\mathbb{L}p_! : \mathsf{Ho}\,\mathsf{Sect}_\mathcal{W}(\Delta\mathcal{C}, \mathcal{M}) \rightleftarrows \mathsf{Ho}\,\mathsf{Sect}(\mathcal{C}, \mathcal{M}) : \mathbb{R}p^*$ is an equivalence.

Since the functor $p : \Delta\mathcal{C} \to \mathcal{C}$ is surjective on objects, we have that the functor $\mathbb{R}p^*$ is conservative. Everything will thus follow from the triangle identities if we verify that the unit map $id \to \mathbb{R}p^*\mathbb{L}p_!$ is an isomorphism. Given a fibrewise-weakly-constant $S \in \mathrm{Ho}\,\mathsf{Sect}_{\mathcal{W}}(\Delta\mathcal{C}, \mathcal{M})$, the unit map evaluated at $\tau : [n] \to \mathcal{C}$ becomes

$$S(\tau) \to \mathbb{L}\varinjlim_{\Delta\mathcal{C}/c} Res_c S|_{\Delta\mathcal{C}/c} \qquad (3.22)$$

where $c = \tau(n)$. The map (*3.22*) is induced by the canonical inclusion $i : [0] \to \Delta\mathcal{C}/c$ picking up $(\tau, \tau(c) = c)$ in the following way. Write as usual $\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}/c, \mathcal{M}(c))$ for the full subcategory of $\mathsf{Fun}(\Delta\mathcal{C}/c, \mathcal{M}(c))$ consisting of $X : \Delta\mathcal{C}/c \to L\mathcal{M}(c)$ that send to $\mathcal{W}(c)$ the maps of $\Delta\mathcal{C}/c$ projecting to identities under the composition $\Delta\mathcal{C}/c \to \Delta\mathcal{C} \to \mathcal{C}$. Denoting also $\delta : \Delta\mathcal{C}/c \to [0]$ the canonical functor, we have the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ho}\,\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}/c, \mathcal{M}(c)) & \overset{\delta^*}{\longleftarrow} & \mathrm{Ho}\,\mathcal{M}(c) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{Ho}\,\mathcal{M}(c) & \underset{=}{\longleftarrow} & \mathrm{Ho}\,\mathcal{M}(c)
\end{array}
\qquad (3.23)
$$

and the map (*3.22*) is the evaluation at $Res_c S|_{\Delta\mathcal{C}/c}$ of the associated Beck-Chevalley map

$$i^* \to \mathbb{L}\delta_! \qquad (3.24)$$

(the homotopy colimit functor $\mathbb{L}\delta_!$ restricts well to the $\mathcal{W}$-subcategory in question). It will suffice to prove the invertibility of (*3.24*). Lemma 3.44 applied to a constant fibration gives an equivalence $L\,\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}/c, \mathcal{M}(c)) \to \mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}/c, L\mathcal{M}(c))$. Thus the diagram (*3.23*) is equivalent in a compatible manner to the Ho -image of the following diagram of infinity-categories:

$$
\begin{array}{ccc}
\mathsf{Fun}_{\mathcal{W}}(\Delta\mathcal{C}/c, L\mathcal{M}(c)) & \overset{p^*}{\longleftarrow} & L\mathcal{M}(c) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle =} \\
L\mathcal{M}(c) & \underset{=}{\longleftarrow} & L\mathcal{M}(c).
\end{array}
\qquad (3.25)
$$

The standard arguments about Beck-Chevalley maps then imply that (*3.24*) is invertible iff the Beck-Chevalley map $i^* \to p_!$ arising from the (higher-categorical) diagram (*3.25*) is an equivalence (we also use that the quasicategory $L\mathcal{M}(c)$ is cocomplete, Corollary 3.41).

It will be thus sufficient to prove the following. For any $X : \Delta\mathcal{C}/c \to L\mathcal{M}(c)$ sending to equivalences those maps of $\Delta\mathcal{C}/c$ that project to identities under the composition $\Delta\mathcal{C}/c \to \Delta\mathcal{C} \to \mathcal{C}$, one has that

$$X(\sigma) \to \varinjlim_{\Delta\mathcal{C}/c} X$$

is an equivalence. This is done exactly as in the proof of Lemma 3.53. $\qquad\square$

A Appendix

A.1 Over the simplex category

In what follows, we will identify partially ordered sets, henceforth referred as posets, with small categories having at most one morphism between each two objects.

Definition A.1. Denote by $[n]$ the category

$$[n] = 0 \to 1 \to 2 \to ... \to n$$

with exactly one morphism from $i \to j$ when $i \leq j$, and no other morphisms. Denote by Δ the full subcategory of **Cat** consisting of categories $[n]$ for $n \geq 0$.

Lemma A.2. *Each morphism in Δ can be factored as a surjection (in the poset sense) followed by an injection (in the poset sense). Surjections and injections form a factorisation system (Δ_s, Δ_i) on Δ which, together with the natural choice of a degree, $\deg[n] = n$, makes it into a Reedy category.*

Proof. Clear. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary A.3. *The category Δ^{op} is a Reedy category for the factorisation system $(\Delta_-^{\mathsf{op}}, \Delta_+^{\mathsf{op}})$ consisting of (the opposites of) injections and surjections.*

Definition A.4. A map $\rho : [m] \to [n]$ of Δ is *a Segal inclusion*, or simply *Segal* iff it is an interval inclusion of $[m]$ as first $m + 1$ elements of $[n]$, i.e. $\rho(i) = i$ for $0 \leq i \leq m$. In particular, m should be less or equal than n.

A map $\zeta : [n] \to [m]$ of Δ is *anchor* iff it preserves the endpoints: $\zeta(n) = m$.

We denote by A, Σ the subcategories of anchor and Segal maps in Δ. It is easy to see that (A, Σ) is a factorisation system on Δ.

Definition A.5. A *Segal factorisation system* on Δ^{op} consists of the pair $(\mathscr{S}, \mathscr{A})$ where $\mathscr{S}$ is the subcategory of Segal maps induced from Σ^{op}, and $\mathscr{A}$ is the subcategory of anchor maps induced from A^{op}.

Lemma A.6. *The identity functor sends Δ_+^{op} to $\mathscr{A}$.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition A.7. A Δ-indexed category is a discrete Grothendieck opfibration $\pi : \mathcal{X} \to \Delta^{\mathsf{op}}$ (that is, every map of $\mathcal{X}$ is π-opcartesian). In particular, there exist a unique, up to isomorphism, simplicial set representing π through the Grothendieck construction.

We shall often write $\mathcal{X}$ instead of π, when this abuse of notation leads to no confusion.

Lemma A.8. *Let $\pi : \mathcal{X} \to \Delta^{\mathsf{op}}$ be a Δ-indexed category. Then*

1. *there is a factorisation system $(\mathfrak{X}_-, \mathfrak{X}_+)$ which π-projects to $(\Delta_-^{\mathsf{op}}, \Delta_+^{\mathsf{op}})$. We call it the Reedy factorisation system of $\mathfrak{X}$.*

2. *There is a factorisation system $(\mathscr{S}_{\mathfrak{X}}, \mathscr{A}_{\mathfrak{X}})$, which π-projects to $(\mathscr{S}, \mathscr{A})$. We call it the Segal factorisation system on $\mathfrak{X}$.*

3. *The identity functor $id : \mathfrak{X} \to \mathfrak{X}$ preserves the maps of the right class: $id(\mathfrak{X}_+) \subset \mathscr{A}_{\mathfrak{X}}$.*

Proof. Immediate. $\square$

Corollary A.9. *Let $\mathcal{E} \to \mathfrak{X}$ be an admissible model semifibration over the Reedy factorisation system $(\mathfrak{X}_-, \mathfrak{X}_+)$, then the category $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$ is a model category.* $\square$

A.2 Normalised model structure

For a variation of the argument of Theorem 2.11, let us look at the following. Let $\mathfrak{X}$ be a Δ-indexed category. The subcategory $\mathfrak{X}_+ \subset \mathscr{A}_{\mathfrak{X}}$ controls degenerations. Recasting the usual definition,

Definition A.10. An object $x \in \mathfrak{X}$ is *degenerate* if there exists a non-identity degree-raising map $y \to x$ of $\mathfrak{X}_+$. An object x is thus non-degenerate iff $\mathfrak{X}_+/x = \{id : x \to x\}$, or, equivalently, $Lat(x) = \emptyset$.

For the purposes of this subsection, consider a functor $\mathcal{E} \to \mathfrak{X}$ such that

1. It is a semifibration over the Segal factorisation system.

2. The induced semifibration over the Reedy factorisation system $(\mathfrak{X}_-, \mathfrak{X}_+)$ is an admissible model Reedy semifibration.

3. It is normalised, that is its restriction $\mathcal{E} \to \mathscr{A}_{\mathfrak{X}}$ is a locally constant fibration, a fibration for which all the transition functors are equivalences.

Given a section $X \in \mathsf{Sect}(\mathfrak{X}, \mathcal{E})$ of such $\mathcal{E} \to \mathfrak{X}$, we can thus conclude that $\mathscr{L}_x X$ is the initial object of $\mathcal{E}(x)$ for each non-degenerate x.

Definition A.11. A section X is *normalised* iff it takes any arrow of $\mathfrak{X}_+$ to an opcartesian arrow of $\mathcal{E} \to \mathfrak{X}$.

Lemma A.12. *A section X is* normalised *iff for any degenerate object $y \in \mathfrak{X}$, the latching map $\mathscr{L}_y X \to X(y)$ is an isomorphism.*

Proof. Given the definition of a normalised section, we have that for each $f : x \to y$ in $\mathfrak{X}_+/y$, the map $f_! X(x) \to X(y)$ is an isomorphism. One then checks that the latching category $Lat(y) \subset \mathfrak{X}_+/y$ is connected and so the colimit of a constant $Lat(y)$-diagram with value $X(y)$ gives $X(y)$. $\qquad\square$

Remark A.13. If we take $x \to y$ to be an ordinary degeneracy (if projected to Δ) then $X(x) \to X(y)$ is an isomorphism (note that $\mathcal{E}(x) \cong \mathcal{E}(y)$.

Denote by $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E}) \subset \mathsf{Sect}(\mathfrak{X}, \mathcal{E})$ the full subcategory of normalised sections.

Lemma A.14. *The category* $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E})$ *admits limits and colimits, which are calculated in* $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$.

Proof. The colimit part is trivial and is left to the reader. For the limit part, we will use the Segal factorisation system on $\mathfrak{X}$ to calculate limits. For the proof, recall also the functor $\pi : \mathfrak{X} \to \Delta^{\mathsf{op}}$.

Let $x \in \mathfrak{X}$, and consider the category $x \backslash \mathscr{S}_{\mathfrak{X}}$. Given that on the level of Δ, the maps of $\mathscr{S}_{\mathfrak{X}}$ are interval inclusions, and so we have an equivalence $x \backslash \mathscr{S}_{\mathfrak{X}} \cong \pi(x) \in \Delta \subset \mathbf{Cat}$.

Now, consider a morphism $f : x \to x'$ in $\mathfrak{X}_+$. It also means that f belongs to $\mathscr{A}_{\mathfrak{X}}$, but in any case, the factorisation system $(\mathscr{S}_{\mathfrak{X}}, \mathscr{A}_{\mathfrak{X}})$ defines a functor $\bar{f} : x' \backslash \mathscr{S}_{\mathfrak{X}} \to x \backslash \mathscr{S}_{\mathfrak{X}}$ by projecting to Δ, one can examine and check that f^*, after the equivalences $x' \backslash \mathscr{S}_{\mathfrak{X}} \cong \pi x'$ and $x \backslash \mathscr{S}_{\mathfrak{X}} \cong \pi x$, is just the map

$$\pi(f) : \pi x' \to \pi x$$

corresponding to f by projection to Δ^{op}. In all, we constructed the following diagram

$$
\begin{array}{ccc}
x' \backslash \mathscr{S}_{\mathfrak{X}} & \xrightarrow{\bar{f}} & x \backslash \mathscr{S}_{\mathfrak{X}} \\
\cong \big\uparrow & & \big\uparrow \cong \\
\pi x' & \xrightarrow{\pi(f)} & \pi x.
\end{array}
$$

If we note by $p_x : x \backslash \mathscr{S}_{\mathfrak{X}} \to \mathfrak{X}$ and $p_{x'} : x' \backslash \mathscr{S}_{\mathfrak{X}}$ the natural projections, then the map $\bar{f}^* p_x^* \mathcal{E} \to p_{x'}^* \mathcal{E}$ (cf Proposition 1.45) of prefibrations over $x' \backslash \mathscr{S}_{\mathfrak{X}}$ is in fact an equivalence due to the normalisation condition, since the natural transformation which induces it, $p_x \bar{f} \to p_{x'}$, lies in $\mathfrak{X}_+$ and not just in $\mathscr{A}_{\mathfrak{X}}$. Hence there is no confusion about lifting $\mathcal{E} \to \mathfrak{X}$ to this diagram. When computing limits in $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$, it is done by taking limits of certain sections over categories like $x \backslash \mathscr{S}_{\mathfrak{X}}$. It will thus suffice to check that the functor

$$\mathsf{Sect}(x \backslash \mathscr{S}_{\mathfrak{X}}, p_x^* \mathcal{E}) \xrightarrow{\bar{f}^*} \mathsf{Sect}(x' \backslash \mathscr{S}_{\mathfrak{X}}, \bar{f}^* p_x^* \mathcal{E}) \cong \mathsf{Sect}(x' \backslash \mathscr{S}_{\mathfrak{X}}, p_{x'}^* \mathcal{E})$$

preserves limits, and the resulting section will then be normalised. But this is equivalent to showing that the functor

$$\pi(f)^* : \mathsf{Sect}(\pi x, \mathcal{E}) \to \mathsf{Sect}(\pi x, \mathcal{E})$$

preserves limits. This is sufficient to test when πf is an elementary degeneracy, and in this case $\pi f : \pi x' \to \pi x$ admits both left and right adjoints. All this suffices to show that, when we compute a limit of a diagram of normalised sections, the values of the limit on degeneracies are isomorphisms. $\qquad\square$

Denote by $\mathfrak{X}_{nd}$ the subcategory of $\mathfrak{X}$ consisting of nondegenerate objects. Clearly, $\mathfrak{X}_{nd} \subset \mathfrak{X}_-$, and moreover it is naturally an inverse Reedy category. Consequently, for each $x \in \mathfrak{X}_{nd}$ and a section $X : \mathcal{C} \to \mathcal{E}$, we can define $\mathscr{M}_x^{nd} X$, the matching object of X at x in the category $\mathsf{Sect}(\mathfrak{X}_{nd}, \mathcal{E})$. It is defined as the limit

$$\mathscr{M}_x^{nd} X = \varprojlim_{Mat^{nd}(x)} R_x X|_{Mat^{nd}(x)}$$

where $Mat^{nd}(x) \subset x\backslash\mathfrak{X}_{nd}$ is the subcategory of all maps out of x in $\mathfrak{X}_{nd}$ safe the identity.

The inclusion $x\backslash\mathfrak{X}_{nd} \subset x\backslash\mathfrak{X}_-$ induces the functor $Mat^{nd}(x) \subset Mat(x)$, and thus a map $\mathscr{M}_x X \to \mathscr{M}_x^{nd} X$.

Lemma A.15. *Let X be a normalised section. Then the map $\mathscr{M}_x X \to \mathscr{M}_x^{nd} X$ is an isomorphism for each $x \in \mathfrak{X}_{nd}$.*

Proof. One has to observe, that in $x\backslash\mathfrak{X}_-$, there are objects $x \to y$ such that y may be degenerate. For such each y, choose a non-degenerate $\bar{y}$ and a map $\bar{y} \to y$ in $\mathfrak{X}_+$ degenerating y. Each such map admits a section $y \to \bar{y}$, which lies in $\mathfrak{X}_-$.

Moreover, if $y \to z$ is a map in $\mathfrak{X}_-$ to a non-degenerate object, there exists a factorisation $y \to \bar{y} \to z$ in $\mathfrak{X}_-$, where $\bar{y}$ is non-degenerate as before. One can see that such observations are sufficient to prove that the functor $Mat^{nd}(x) \to Mat(x)$ is final (or right cofinal in the sense of [12]), and this implies the isomorphism of limits. $\qquad\square$

Theorem A.16. *The category $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E})$ possesses a model structure with limits and colimits created by the inclusion to $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$. The classes of cofibrations, fibrations and weak equivalences are given as follows:*

- *a map $A \to B$ of $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E})$ is a cofibration iff it is a Reedy cofibration in $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$,*

- *a map $A \to B$ of $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E})$ is a weak equivalence iff it is such in $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$,*

- *a map $X \to Y$ of $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E})$ is a fibration iff for each non-degenerate object $x \in \mathfrak{X}$, the relative matching map $X(x) \to Y(x) \prod_{\mathscr{M}_x Y} \mathscr{M}_x X$ is a fibration in $\mathcal{E}(x)$.*

Moreover $\mathscr{M}_x X \cong \mathscr{M}_x^{nd} X$ for each nondegenerate $x \in \mathfrak{X}_{nd}$.

Lemma A.17. *In $\mathsf{Sect}_N(\mathfrak{X}, \mathcal{E})$,*

- *a map $A \to B$ is a cofibration and a weak equivalence iff for each $x \in \mathfrak{X}$, the relative latching map $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a trivial cofibration in $\mathcal{E}(x)$,*

87

- *a map $X \to Y$ is a fibration and a weak equivalence iff for each non-degenerate object $x \in \mathfrak{X}$, the relative matching map $X(x) \to Y(x) \prod_{\mathcal{M}_x Y} \mathcal{M}_x X$ is a trivial fibration in $\mathcal{E}(x)$.*

Proof. The first is done by restricting to $\mathsf{Sect}(\mathfrak{X}_+, \mathcal{E})$ and using Corollary 2.21, just as for Lemma 2.23. The second is done by restricting to $\mathsf{Sect}(\mathfrak{X}_{nd}, \mathcal{E})$, and using the dual of Corollary 2.21 together with Lemma A.15. $\qquad\square$

Proof of Theorem A.16.

1. The limits and colimits axiom is clear, see Lemma A.14.

2. The weak equivalences of $\mathsf{Sect}(\mathfrak{X}, \mathcal{E})$ satisfy three-for-two, hence the same property applies for the weak equivalences between non-degenerate sections.

3. The retract stability for the three classes of maps is verified just as in Lemma 2.14.

4. The lifting is proven analogously to the Reedy case. Consider a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & S \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & T
\end{array}
$$

   with, say, f a cofibration and p a trivial fibration, and we keep in mind the result of Lemma A.17. We observe that each degree zero object x of $\mathfrak{X}$ has empty latching and matching categories, and is moreover non-degenerate. Hence in this case the relative latching map reduces to a cofibration $A(x) \to B(x)$, the relative matching map reduces to a trivial fibration $S(x) \to T(x)$, and finding a lifting is trivial. For the induction step, consider, for a non-degenerate $x \in \mathfrak{X}_{nd}$, the diagram

$$
\begin{array}{ccccc}
A(x) & \longrightarrow & A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B & \longrightarrow & S(x) \\
& \searrow & \downarrow & & \downarrow \hspace{0.5em} \searrow \\
& & B(x) & \longrightarrow & T(x) \coprod_{\mathscr{M}_x T} \mathscr{M}_x S \longrightarrow T(x).
\end{array}
$$

   which admits a lifting as in Reedy case. If $y \in \mathfrak{X}$ is, however, a degenerate object, then $\mathscr{L}_y A \cong A(x)$ and $\mathscr{L}_y B \cong B(x)$, and the square

$$
\begin{array}{ccc}
\mathscr{L}_y A & \overset{\sim}{\longrightarrow} & A(y) \\
\downarrow & & \downarrow{\scriptstyle f(y)} \\
\mathscr{L}_y B & \overset{\sim}{\longrightarrow} & B(y)
\end{array}
$$

is a pushout, hence the relative latching map is isomorphic to $B(y) \to B(y)$, and finding the lift in

$$
\begin{array}{ccccc}
A(y) & \longrightarrow & B(y) & \longrightarrow & S(y) \\
& \searrow & \downarrow = & & \downarrow \\
& & B(y) & \longrightarrow & T(y) \coprod_{\mathscr{M}_y T} \mathscr{M}_y S \longrightarrow T(y).
\end{array}
$$

is trivial, whichever the property the map on the right of the square possesses.

5. Assume given a map of normalised section $A \to C$. Degree zero objects x are non-degenerate and have no matching-latching categories, so we simply factor our map as $A(x) \to B(x) \to C(x)$ using the model structure of $\mathcal{E}(x)$. So far, B is trivially a normalised section.

By induction, we have constructed the factorisation $A(y) \to B(y) \to C(y)$ for objects $y \in \mathfrak{X}$ of degree less than n, and $B : \mathfrak{X}_{<n} \to \mathcal{E}$ is non-degenerate. For x of degree n, there is the following diagram

$$
\begin{array}{ccccc}
\mathscr{L}_x A & \longrightarrow & A(x) & \longrightarrow & \mathscr{M}_x B \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{L}_x B & \longrightarrow & C(x) & \longrightarrow & \mathscr{M}_x C
\end{array}
$$

which exists due to the inductive assumption and provides us with the following map

$$
\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.
$$

If x is non-degenerate, we factor it as

$$
\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.
$$

which, together with maps $\mathscr{L}_x B \to B(x)$ and $B(x) \to \mathscr{M}_x B$, yields the desired extension of the factorisation to x. For a degenerate object y, we simply put $B(y) = \mathscr{L}_y B \coprod_{\mathscr{L}_y A} A(y)$. Then the natural map $\mathscr{L}_y B \to B(y)$ is an isomorphism (since A is normalised) and the factorisation

$$
\mathscr{L}_y B \coprod_{\mathscr{L}_y A} A(y) = B(y) \to C(y) \prod_{\mathscr{M}_y C} \mathscr{M}_y B.
$$

is as needed, given the first map satisfies lifting along any map of $\mathcal{E}(y)$ and the second map is not forced to any condition. $\qquad\square$

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