Lectures for MAA 306 course
“Topics in Differential Geometry”

Edouard Balzin

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What is geometry?

Is it topology? Yes. But is all topology geometry? Not quite.

Is it algebra? Yes. It is a lot of algebra, in fact (all "commutative algebra" and even some noncommutative algebra in case of algebraic geometry).

Is it analysis? Often, yes. Even algebraically defined varieties can be studied using analytic techniques.

Because of this, the course might have a different type of difficulty: many things coming together at once.

Goal: introduce smooth manifolds, main objects that come together (smooth maps, tensor fields), and prove a few results. By no means a complete course of differential geometry (that takes $3\times$ more hours!).

Still, I hope it could give you some taste for your masters.

1 Topology and topological manifolds

1.1 Topological spaces

Topological spaces and maps Top

Definition 1.1. A *Topological space* consists of a set X together with a choice of subsets $\mathsf{Op}(X) \subset 2^X$. A subset $U \in \mathsf{Op}(X)$ is called *open*. The opens satisfy:

1. $\emptyset \in \mathsf{Op}(X)$ and $X \in \mathsf{Op}(X)$,
2. Given $U_1, ...U_n$ in $\mathsf{Op}(X)$, their intersection $U_1 \cap ... \cap U_n \in \mathsf{Op}(X)$,
3. Given any family $\{U_i\}_{i\in I}$ of opens (the set I can be infinite), one has $\cup_{i\in I} U_i \in \mathsf{Op}(X)$.

One can have different choices of $\mathsf{Op}(X)$ for the same set X. We shall often omit $\mathsf{Op}(X)$ from the notation.

Definition 1.2. Let $(X, \mathsf{Op}(X))$ and $(Y, \mathsf{Op}(Y))$ be two topological spaces. A *continuous map* from X to Y is a function $f : X \to Y$ such that $U \in \mathsf{Op}(Y)$ implies $f^{-1}(U) := \{x \in X | f(x) \in U\} \in \mathsf{Op}(X)$. The map f is furthermore a *homeomorphism* if it admits a continuous inverse $g : Y \to X$: $f \circ g = \mathsf{id}_Y, g \circ f = \mathsf{id}_X$. We denote $f^{-1} := g$.

A homeomorphism $f : X \xrightarrow{\sim} Y$ is in particular an open map: if $U \in \mathsf{Op}(X)$ then $f(U) = (f^{-1})^{-1}(U) \in \mathsf{Op}(Y)$.

I will sometimes write $X \in \mathsf{Top}$ and $f \in \mathsf{Top}$ to mean that X is a topological space and that f is continuous.

Examples in Top

Example 1.3. Take X any set and put $\mathsf{Op}(X) := 2^X$. It works. This is called *discrete topology* on X. Any singleton $\{x\}$ is indeed open in this topology.

Let Y be a space. Any function $X \to Y$ is continuous for discrete topology on X (what about $Y \to X$?)

Example 1.4. Take $X = \mathbb{R}$. Put $\mathsf{Op}(X)$ to be all U such that $x \in U \Rightarrow \exists \varepsilon > 0$ such that $]x - \varepsilon, x + \varepsilon[\subset U$. This is your classic analysis course topology.

1. Nothing to try for $\emptyset$, trivial for X,
2. Take $\varepsilon = \min(\varepsilon_1, ... \varepsilon_n)$
3. $x \in \cup_I U_i$ means that $x \in U_j$ for some $j \in I$, then use that $]x - \varepsilon, x + \varepsilon[\subset U_j \subset \cup_I U_i$.

For $\mathbb{R}^n$, the definition is repeated: $x \in U \Rightarrow \exists \varepsilon > 0$ such that $B(x, \varepsilon) := \{y | \, \|y - x\| < \varepsilon\} \subset U$. Can also use open cubes.

The notion of continuous function $f : \mathbb{R}^n \to \mathbb{R}^k$ coincides with that of ε-δ-continuity.

Closed sets

Definition 1.5. Let X be a topological space. $V \subset X$ is *closed* iff $X \setminus V$ is open. Closed sets satisfy:

1. $\emptyset, X$ are closed.
2. If $V_1, ..., V_k$ are closed then so is $V_1 \cup ... \cup V_k$.
3. The intersection $\cap_I V_i$ of any family $\{V_i\}_{i \in I}$ is closed.

Example 1.6. Take $X = \mathbb{R}$. Declare $\mathsf{Op}(X)$ by $U \in \mathsf{Op}(X) \iff X \setminus U$ is finite or the whole X. To verify that this gives topology, it is enough to verify 1.-3. from above. 1. is immediate, 2. is true since finite unions of finite subsets are finite, and 3. is true because $|\cap_I V_i| \leq |V_i| < \infty$. This example is called *Zariski topology* on $\mathbb{R}$.

Lemma 1.7. *Let $f : X \to Y$ be a function. Then f is continuous $\iff$ $f^{-1}(V)$ is closed for any closed set $V \subset Y$.*

Proof. For the $\Leftarrow$ direction: U open in $Y \iff Y \setminus U$ closed $\Rightarrow f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ closed $\iff f^{-1}(U)$ open in X. The other part is done similarly. $\square$

Induced topology

Definition 1.8. Let $X \in \mathsf{Top}$ and $S \subset X$ any subset. Define $\mathsf{Op}(S)$ to consist of all the subsets $V \subset S$ that can be presented as $V = U \cap S$ where $U \in \mathsf{Op}(X)$. One verifies that this gives a topology on S, called the *induced topology*.

If $f : X \to Y$ continuous, then so is $f|_S : S \to Y$. If $S \in \mathsf{Op}(X)$, then $\mathsf{Op}(S) = \{U \in \mathsf{Op}(X) | U \subset S\}$.

Example 1.9. The n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is defined as

$$\mathbb{S}^n := \{x = (x_0, ..., x_n) | x_0^2 + ... + x_n^2 = \|x\|^2 = 1\}.$$

Its filling is the unit $n+1$-ball

$$\overline{\mathbb{D}}^{n+1} := \{x = (x_0, ..., x_n) | x_0^2 + ... + x_n^2 = \|x\|^2 \leq 1\}$$

and the interior is

$$\mathbb{D}^{n+1} := \{x = (x_0, ..., x_n) | x_0^2 + ... + x_n^2 = \|x\|^2 < 1\} = B(0, 1).$$

One can consider other zero sets of continuous and smooth functions, or some other subsets, like the torus in $\mathbb{R}^3$. The latter can also be defined without presenting it as a 3-dimensional subset.

Quotient topology

Consider a set X and let $\sim$ be an equivalence relation: $x \sim x$, $x \sim y \iff y \sim x$, and $x \sim y, y \sim z \Rightarrow x \sim z$. Then we can form the quotient set (set of equivalence classes)

$$X/_\sim = \{[x] \mid [x] = \{y | y \sim x\}\}$$

so that $[x] = [y]$ iff $x \sim y$. There is a function $p : X \to X/_\sim$, $x \mapsto [x]$.

The function p satisfies the following property: let $f : X \to Y$ be any function such that $x \sim x' \Rightarrow f(x) = f(x')$. Then there exists unique $\bar{f} : X/_\sim \to Y$ such that $f = \bar{f} \circ p$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}.$$

Indeed, take $\bar{f}([x]) = f(x)$, the result does not depend on the choice of representative since $[x] = [y]$ then $x \sim y$ and $f(x) = f(y)$. Any other map $g : X/_\sim \to Y$ such that $g \circ p = f$ will forcefully satisfy

$$g([x]) = g(p(x)) = f(x) = \bar{f}([x]).$$

Let now $X \in \mathsf{Top}$.

Definition 1.10. The *quotient topology* on $X/_\sim$ is defined as follows: $U \in \mathsf{Op}(X/_\sim)$ iff $p^{-1}(U) \in \mathsf{Op}(X)$.

Why? First, because it works:

Lemma 1.11. *Let $X \in \mathsf{Top}$ and $f : X \to S$ be any map of sets. Define $\mathsf{Op}(S) := \{U \subset S | f^{-1}(U) \in \mathsf{Op}(X)\}$. Then $(S, \mathsf{Op}(S)) \in \mathsf{Top}$ and f is automatically continuous.*

Proof.

1. $f^{-1}(S) = X$, $f^{-1}(\emptyset) = \emptyset$.
2. $f^{-1}(U_1 \cap U_2) = \{x | f(x) \in U_1 \text{ and } f(x) \in U_2\} = f^{-1}(U_1) \cap f^{-1}(U_2) \in \mathsf{Op}(X)$.
3. $f^{-1}(\cup_I U_i) = \{x | f(x) \in U_i \text{ for some } i \in I\} = \cup_I f^{-1}(U_i) \in \mathsf{Op}(X)$. □

Second, because it solves the same universal problem as before. If $f : X \to Y$ is any continuous map satisfying $x \sim y \Rightarrow f(x) = f(y)$, then recall the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}.$$

Take $U \in \mathsf{Op}(Y)$. Then $\bar{f}^{-1}(U)$ is in fact open, as

$$p^{-1}(\bar{f}^{-1}(U)) = \{x \,|\, p(x) \in \bar{f}^{-1}(U)\} = \{x \,|\, f(x) = \bar{f}(p(x)) \in U\} = f^{-1}(U).$$

We summarise it as follows:

Lemma 1.12. *Let $X \in \mathsf{Top}$ and $\sim$ an equivalence relation on X. Then the quotient set map $X \to X/\sim$ satisfies the following property: for any continuous $f : X \to Y$ such that $x \sim y \Rightarrow f(x) = f(y)$, there exists unique continuous map $\bar{f} : X/\sim \to Y$ such that $\bar{f} \circ p = f$.*

The torus $\mathbb{T}$

Example 1.13. Consider the following equivalence relation on $\mathbb{R}^2$:

$$(x, y) \sim (x', y') \iff (x - x', y - y') \in \mathbb{Z}^2.$$

Take the quotient $\mathbb{R}^2/_\sim$ that we will also denote $\mathbb{R}^2/\mathbb{Z}^2$. Another way to present the same set (up to iso): take $[0, 1]^2$ and put $\sim$ on it by declaring

$$(x, 0) \sim (x, 1), (0, y) \sim (1, y) \text{ and } (x, y) \sim (x, y).$$

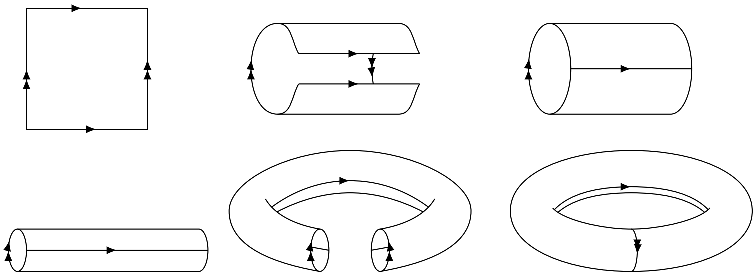
Denote $\mathbb{T} := [0, 1]^2/_\sim$. There are maps $i : [0, 1]^2 \hookrightarrow \mathbb{R}^2$ (inclusion) and $r : \mathbb{R}^2 \to [0, 1]^2$, $r(x, y) = (\{x\}, \{y\})$ (fractional part). These functions respect the equivalences relations inducing bijections

$$\bar{i} : \mathbb{T} \cong \mathbb{R}^2/\mathbb{Z}^2 : \bar{r}$$

The set $\mathbb{T}$ or equivalently $\mathbb{R}^2/\mathbb{Z}^2$ is called the (2-)torus.

Exercise 1.14. *We can use quotient topology and define topology on $\mathbb{R}^2/\mathbb{Z}^2$ taking the standard topology on $\mathbb{R}^2$. We can do the same for $\mathbb{T}$ using the induced topology on $[0, 1]^2$. Prove that for this choice of topologies, the maps $\bar{i}$ and $\bar{r}$ are continuous (hint: Lemma 1.12).*

Checks like this one are abundant and many of them will be skipped.



Real projective spaces $\mathbb{RP}^n$

Important class of spaces that does not naturally come as a subset of $\mathbb{R}^N$.

Example 1.15. The real projective space $\mathbb{RP}^n$ is defined as the set of all one-dimensional subspaces of $\mathbb{R}^{n+1}$:

$$\mathbb{RP}^n := \{\mathcal{L} \subset \mathbb{R}^{n+1} | \mathcal{L} \text{ is linear and } \dim \mathcal{L} = 1\}.$$

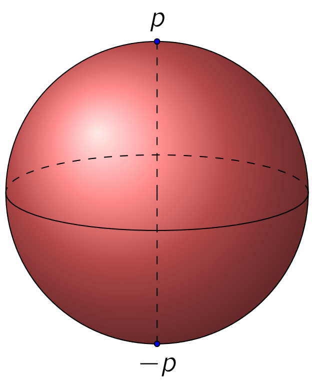
Alternatively we can understand $\mathcal{L} \in \mathbb{RP}^n$ as lines passing through the origin. There are different ways to parametrise these lines.

Take $(x_0, \ldots x_n) \in \mathcal{L} \setminus 0 \subset \mathbb{R}^n$ (0 denotes the zero subspace). Then every other point of $\mathcal{L}$ can be obtained as $(\lambda x_0, \ldots, \lambda x_n)$ where we vary $\lambda \in \mathbb{R}$. This motivates to consider $\mathbb{R}^{n+1} \setminus 0$ and put $\sim$ by declaring $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for all $\lambda \in \mathbb{R}^*$.

We can then consider the map $q : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ that sends $(x_0, \ldots, x_n)$ to the line $\mathcal{L} = \{(\lambda x_0, \ldots, \lambda x_n) | \lambda \in \mathbb{R}\}$. This induces $\bar{q} : (\mathbb{R}^{n+1} \setminus 0)/_\sim \to \mathbb{RP}^n$. It admits a bijective inverse $g : \mathbb{RP}^n \to (\mathbb{R}^{n+1} \setminus 0)/_\sim$ that sends $\mathcal{L}$ to $[(y_0, \ldots, y_n)]$ where $(y_0, \ldots, y_n)$ is a nonzero vector of $\mathcal{L}$.

Use this bijection to put topology on $\mathbb{RP}^n$: $U \subset \mathbb{RP}^n$ is open iff $\bar{q}^{-1}(U) = g(U)$ is open in $\mathbb{R}^{n+1}/_\sim$.

Was that unclear? Go to TD! You will also learn why $\mathbb{RP}^n \cong \mathbb{S}^n/_\sim$, where the equivalence relation identifies antipodal points.



$$\mathbb{RP}^n \cong \mathbb{S}^n/_\sim, \ p \sim -p.$$

1.2 Topological manifolds

Topological manifolds

Definition 1.16. A topological space X is *Hausdorff* if for $x, y \in X$, $x \neq y$ there exist two opens $U_x \ni x$, $U_y \ni y$ such that $U_x \cap U_y = \emptyset$.

Example 1.17. The line $\mathbb{R}$ with Zariski topology is **not** Hausdorff.

Definition 1.18. A *topological n-manifold* is a Hausdorff topological space M such that for any point $x \in M$ there exists $U \in \mathsf{Op}(M)$ containing x and a homeomorphism between U and an open subset of $\mathbb{R}^n$. The number n is called the *dimension* of M.

The definition relies on the fact that opens of $\mathbb{R}^n$ are not homeomorphic to opens of $\mathbb{R}^m$ for $n \neq m$. This is a not too obvious result that we will avoid (smooth manifolds have it easy).

Example 1.19. Any $\mathbb{R}^n$ with standard topology is a topological n-manifold. The unit open disk $\mathbb{D}^n \subset \mathbb{R}^n$ is a topological n-manifold. Any open $U \subset \mathbb{R}^n$ is also a topological n-manifold. Example: $\mathsf{GL}(n, \mathbb{R}) := \{M \in \mathsf{Mat}_n(\mathbb{R}) | \det M \neq 0\} \subset \mathbb{R}^{n^2}$ of dimension n^2.

$\mathbb{S}^n$ is a topological manifold

Example 1.20. The spheres $\mathbb{S}^n$. Define

$$U_i^{\pm} = \{(x_0, ..., x_n) \in \mathbb{S}^n | \pm x_i > 0\}$$

($2n+2$ sets). One has $\left(\cup_{i=0}^n U_i^+\right) \cup \left(\cup_{i=0}^n U_i^-\right) = \mathbb{S}^n$. Each $U_i^{\pm}$ is open as it is equal to $\mathbb{S}^n \cap \mathbb{H}_i^{\pm}$ where $\mathbb{H}_i^{\pm} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid \pm x_i > 0\}$.

We then look at the map

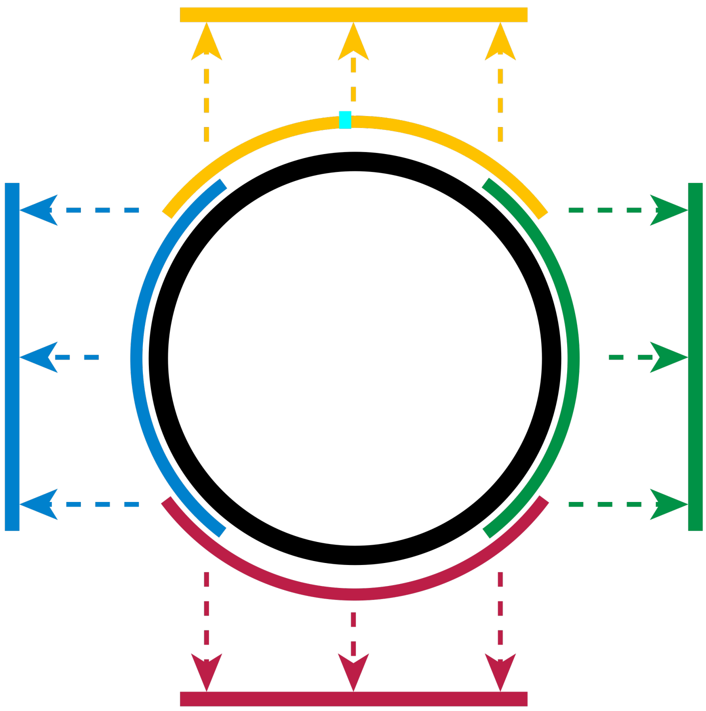
$$\varphi_i^{\pm} : U_i^{\pm} \to \mathbb{D}^n \subset \mathbb{R}^n, \ \ \varphi_i^{\pm}(x_0, ..., x_n) = (x_0, ..., \hat{x}_i, ..., x_n),$$

- it goes to the unit disk $\mathbb{D}^n$ as $x_0^2 + ... + x_n^2 = 1$ in $U_i^{\pm}$ implies $x_0^2 + ... + \widehat{x_i^2} + ... + x_n^2 < 1$
- it is continuous being a restriction of the i-th projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$ (linear).
- it is a homeomorphism, with inverse given by

$$\eta_i^{\pm} : \mathbb{D}^n \to U_i^{\pm}, \ \ \eta_i^{\pm}(y_1, ..., y_n) = (y_1, ..., y_{i-1}, \pm\sqrt{1 - \sum_{j=1}^n y_j^2}, y_i, ..., y_n);$$

  $\eta_i^{\pm}$ is continuous when viewed as map $\mathbb{D}^n \to \mathbb{H}_i^{\pm}$, its image is exactly $U_i^{\pm}$, composing $\eta_i^{\pm}$ with $\varphi_i^{\pm}$ either way gives identities.

Image courtesy of Wikipedia



There are of course other ways to do charts (try stereographic projection for $\mathbb{S}^1$ and $\mathbb{S}^2$).

$\mathbb{RP}^n$ is a topological manifold

Example 1.21. Recall $q : \mathbb{R}^{n+1} \setminus 0 \twoheadrightarrow \mathbb{RP}^n$, $v \mapsto \mathsf{Span}(v)$. This presents $\mathbb{RP}^n$ as a quotient of $\mathbb{R}^{n+1} \setminus 0$ by the equivalence relation $v \sim \lambda v$ for $\lambda \neq 0$. Take a line $\mathcal{L} \in \mathbb{RP}^n$, $\mathcal{L} = \mathsf{Span}(e)$ for some $e \neq 0$. Denote $H := \mathcal{L}^{\perp}$. As we studied, $\mathbb{R}^{n+1} = \mathsf{Span}(e) \oplus H$. Denote $C := e + H$.

Let $\mathcal{L}' = \mathsf{Span}(v)$ be any other line. We would like to "project" onto C, which corresponds to taking $C \cap \mathcal{L}'$. In terms of $\mathbb{R}^{n+1}$, this corresponds to a map

$$p_C : \mathbb{R}^{n+1} \setminus H \to C, \quad v = ae + h \mapsto e + h/a;$$

indeed, for this to make sense, one needs $a \neq 0 \Leftrightarrow \mathcal{L}' \notin H$. The map p_C is continuous with image C. It also respects the equivalence relation: if $v = ae + h$ then

$$\lambda v = \lambda ae + \lambda h \;\mapsto\; e + \lambda h/a\lambda = e + h/a.$$

The inclusion $i_C : C \hookrightarrow \mathbb{R}^{n+1} \setminus H$ works as $e + h \mapsto e + h$ so $p_C i_C = \mathsf{id}_C$. Note that $i_C(p_C(v)) \sim v$ (take $\lambda = 1/a$).

Upshot: have continuous $p_C : \mathbb{R}^{n+1} \setminus H \to C$, and have $i_C : C \to \mathbb{R}^{n+1} \setminus H$ that respect $\sim$. Denote $P_C := \bar{p}_C : \mathbb{RP}^n \setminus q(H) \to C$ and $I_C := q \circ i_C : C \to \mathbb{RP}^n \setminus q(H)$. Both are continuous and are mutually inverse. We conclude by choosing a basis in H that gives a homeomorphism $\mathbb{R}^n \cong C$.

1.3 Leftover statements

Here I will add some statements that were mentioned in course and can be of use for later.

Lemma 1.22. *Let $f : X \to Y$ be a continuous map. Denote $S = \operatorname{im} f \subset Y$ and equip it with induced topology. Then f viewed as a map $X \to S$ is still continuous.*

Proof. By definition of the image, for each $V \in \mathsf{Op}\, Y$ one has $f^{-1}(V) = f^{-1}(V \cap S)$. We can read this equality from right to left and conclude. $\square$

2 Smooth manifolds

Differential calculus

Aubin book [3], Chapter 0 is good review.

Choose U open in $\mathbb{R}^n$. A function $f : U \to \mathbb{R}^m$ is *differentiable at* $p \in U$ if there exists a (forcefully unique) linear map $df(p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that for all $h \in \mathbb{R}^n$ sufficiently close to 0,

$$f(p+h) = f(p) + df(p)(h) + \|h\|\, \varepsilon(p, h)$$

with $\varepsilon(p, h) \to 0$ as $h \to 0$. A differentiable f is automatically continuous at p.

In standard coordinates $p = (x_1, \ldots, x_n)$ the differential $df(p)$ is computed via partial derivatives (Jacobian matrix):

$$J = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{j=1,n}^{i=1,m} \in \mathsf{Mat}_{m,n}(\mathbb{R}) \text{ so that } df(h)_i = \sum_j h_j \frac{\partial f_i}{\partial x_j}(p).$$

In coordinate-independent terms, instead of partial derivatives one speaks of directional derivative: for $v \in \mathbb{R}^n$, one defines

$$D_v f(p) := \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}.$$

Then $D_v f(p) \in \mathbb{R}^m$ and can be computed in local coordinates as $(\sum v_j \frac{\partial f_i}{\partial x_j}(p))_i$. Note that v belongs to $\mathbb{R}^n$ and not to U. This difference will become quite pronounced in our course.

When the assignment $df : p \mapsto df(p)$ gives a continuous map $\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ we say that f is of class C^1. Passing between differential and directional derivatives is not a problem in this class of functions (otherwise directional derivability does not imply full derivability).

The chain rule is summarised very neatly: let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ both open, and $f : U \to V$, $g : V \to W \subset \mathbb{R}^k$ of class C^1. then $d(g \circ f) = dg \circ df$. This way it is easy to see that in no way U and V can be C^1-isomorphic, or *diffeomorphic*, if $m \neq n$: having such an isomorphism $f : U \cong V : f^{-1}$ would imply that $df : \mathbb{R}^n \cong \mathbb{R}^m : d(f^{-1})$ is a vector space isomorphism.

One can further define C^k (f is of class C^k if df is of class C^{k-1} with $C^0 = C$) and C^∞ (f is of class C^k for all k). The notation $C^k(U, V)$ for $k \in [0, \infty]$ means the set of all functions $U \to V$ of class C^k, and $C^k(U) := C^k(U, \mathbb{R})$. If $f \in C^k(U, V)$ and $g \in C^k(V, W)$ then $g \circ f \in C^k(U, W)$.

The set $C^k(U)$ is, in particular, a commutative $\mathbb{R}$-algebra. The vector space structure is $(\lambda f)(p) := \lambda f(p)$, $(f + g)(p) := f(p) + g(p)$ and the ring structure is $f \cdot g(p) := f(p)g(p)$ (product is differentiable thanks to Leibniz rule).

2.1 Attempting to define smooth manifolds

Definition 2.1. Let M be a topological n-manifold and $p \in M$. A *coordinate chart* is a pair (U, φ) of an open $U \subset M$ and a homeomorphism $\varphi : U \xrightarrow{\sim} \Omega$ onto an open $\Omega \subset \mathbb{R}^n$. The (coordinate) chart is *centered* at p if $p \in U$ and $\varphi(p) = 0$.

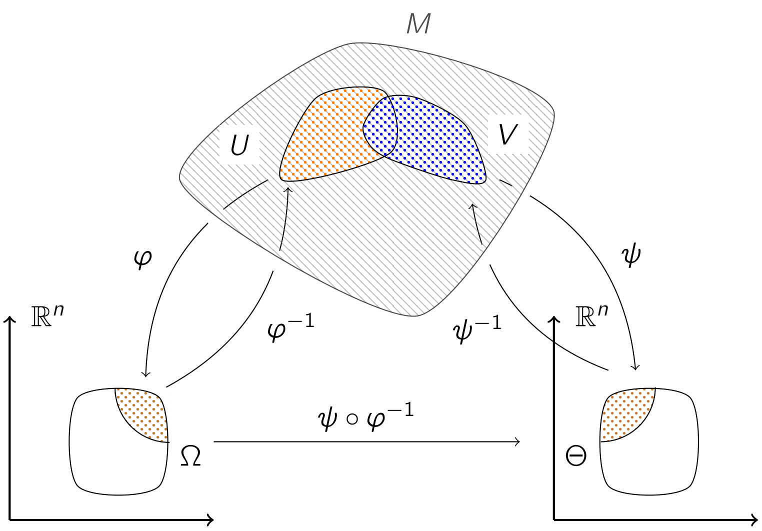
This definition can be given with M a topological space, just that nothing will guarantee the existence of charts.

Let M be a topological manifold. We certainly have $C(M)$, the set of continuous functions $M \to \mathbb{R}$. We can also consider $C(M, N)$, continuous maps to other topological manifolds N (or even spaces).

Taking $U \subset M$ and choosing $\varphi : U \xrightarrow{\sim} \Omega$ with Ω open in $\mathbb{R}^n$ might make us think that we can define $C^k(U)$ by saying $f \in C^k(U)$ iff $f \circ \varphi^{-1} \in C^k(\Omega)$. The problem is that we have no fail-safe in Definition 1.18 that makes it independent of φ.

Put another way, if $p \in M$ and (U, φ), (V, ψ) are two coordinate charts containing p, we can ask what happens on $U \cap V$. Denote $\Phi = \varphi(U \cap V)$ and $\Psi = \psi(U \cap V)$. Then the map $\psi \circ \varphi^{-1}|_\Phi : \Phi \to \Psi$ is a continuous map between two subsets of $\mathbb{R}^n$, a homeomorphism, but not necessarily $C^{\geq 1}$.

There are examples of topological manifolds which cannot be "promoted" to smooth manifolds (E_8 manifold).



Atlas

Let M be a Hausdorff topological space. Assume we have two open charts (U, φ), (V, ψ) with $\varphi : U \xrightarrow{\sim} \Omega \subset \mathbb{R}^n$, $\psi : V \xrightarrow{\sim} \Theta \subset \mathbb{R}^n$. Denote as before $\Phi = \varphi(U \cap V)$, $\Psi = \psi(U \cap V)$.

Definition 2.2. In the situation above, two charts are called C^k-*compatible* if the map $\psi \circ \varphi^{-1}|_\Phi : \Phi \to \Psi$ is a C^k-diffeomorphism of open sets in $\mathbb{R}^n$.

Note that the inverse of $\psi \circ \varphi^{-1}|_\Phi$, given by $\varphi \circ \psi^{-1}|_\Psi$ is also then a C^k-diffeomorphism.

Definition 2.3. A C^k-*atlas* (of dimension n) on a Hausdorff topological space M consists of a collection of open charts $\{(U_i, \varphi_i\}|_{i\in I}$ with $\varphi_i : U_i \xrightarrow{\sim} \Omega_i$ homeomorphisms onto opens $\Omega_i \subset \mathbb{R}^n$ such that

1. (cover) $\cup_I U_i = M$
2. (compatible) any two charts (U_i, φ_i), (U_j, φ_j) are C^k-compatible.

Lemma 2.4. *Let M be a Hausdorff topological space. Then M is a topological manifold $\Leftrightarrow$ there exists a C^0-atlas on M (of some dimension).* □

Still not quite a smooth manifold

One would be motivated to say that we now define a smooth manifold as a Hausdorff topological space/manifold admitting a C^∞-atlas. Atlases are indeed useful, however such a definition is highly non-invariant, like a vector space with a fixed basis.

Given a Hausdorff M and some smooth atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ we can declare the set of smooth functions $C^\infty(M)$ to consist of those (continuous) $f : M \to \mathbb{R}$ such that

$$\Omega_i \xrightarrow{\varphi_i^{-1}} U_i \xrightarrow{f|_{U_i}} \mathbb{R}$$

is in $C^\infty(\Omega_i)$. This turns out to be a good definition, but different smooth atlases give the same set of smooth functions $C^\infty(M)$.

Example 2.5. $M = \mathbb{R}^n$, $\mathcal{A}_1 = \{(\mathbb{R}^n, \mathsf{id}_{\mathbb{R}^n})\}$, $\mathcal{A}_2 = \{(B(x, 1), \mathsf{id}_{B(x,1)})\}_{x\in\mathbb{Q}^n}$.

One way to remedy this is to define some notion of equivalence relations on atlases. Another way is to properly spell out the idea that $C^\infty(M)$ is glued from locally defined smooth functions.

We will do the latter.

2.2 Controlling local properties of functions

Sheaves of functions

The following exposition exists in [2], a more advanced version can also be found in [5]. Notation: X a space, T a set, $\mathcal{F}(X, T)$ set of all functions. If T has topology then $C(X, T) = \mathsf{Map}(X, T)$ denotes continuous maps as usual.

(If it helps for now, think of X, T as subsets in $\mathbb{R}^n$)

Exercise 2.6. *For what topology on a set T one has $C(X, T) = \mathcal{F}(X, T)$?*

Definition 2.7. Let $X \in \mathsf{Top}$.

1. A *presheaf of functions* valued in T is a collection of subsets $\mathcal{P}(U) \subset \mathcal{F}(U, T)$ for each $U \in \mathsf{Op}(X)$ that satisfies the following property: for $V \subset U$, if $f \in \mathcal{P}(U)$ then $f|_V \in \mathcal{P}(V)$. We will refer to it as $\mathcal{P} = \{\mathcal{P}(U)\}_{U\in\mathsf{Op}(X)}$.
2. A presheaf of functions $\mathcal{P}$ is a *sheaf* if for any $U \in \mathsf{Op}(X)$ and any open cover $\{U_i\}_I$ of U, $U = \cup_I U_i$, one has

$$f \in \mathcal{F}(U, T),\ f|_{U_i} \in \mathcal{P}(U_i) \text{ for all } i \in I \implies f \in \mathcal{P}(U).$$

There are more general definitions of sheaves that we will try to avoid.

Example 2.8. 1. Take X, T any and set $\mathcal{P}(U) = \mathcal{F}(U, T)$ trivially produces a sheaf.

2. Take X, T any and set $\mathcal{P}(U)$ to be constant functions $U \to T$. It is a presheaf, not generally a sheaf (*why?*).

3. Take X any, T a topological space, and define $\mathcal{P}$ by setting $\mathcal{P}(U) = C(U, T)$. It is a presheaf as restrictions of continuous maps are continuous. It is also a sheaf.

   To check, let $f : U \to T$ and $\{U_i\}_I$ cover of U such that $f|_{U_i} \in C(U_i, T)$. Let $V \in \mathsf{Op}(T)$. Take $f^{-1}(V)$. We have that

   $$f^{-1}(V) \cap U_i = f|_{U_i}^{-1}(V) \in \mathsf{Op}(U_i) \subset \mathsf{Op}(U), \quad f^{-1}(V) = \cup_I (f^{-1}(V) \cap U_i)$$

   and thus $f^{-1}(V) \in \mathsf{Op}(U)$ and hence $f \in \mathcal{P}(U)$.

4. In particular, taking T a set and putting on it discrete topology produces $\mathcal{P}(U)$ consisting of locally constant functions $U \to T$.

5. Take X any, $T = \mathbb{R}$ and set

   $$\mathcal{P}(U) = \{f : U \to \mathbb{R} | \exists M_f > 0 : \forall x \in U \ \ \|f(x)\| < M_f\}.$$

   it is a presheaf, not a sheaf. Simply take $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$: locally bounded, globally unbounded.

Example 2.9. Let $X = \Omega \in \mathsf{Op}(\mathbb{R}^n)$, $T = \mathbb{R}$ with standard topology. A function $f : \Omega \to \mathbb{R}$ is differentiable at x iff the same is true for the function $f|_U$, where $x \in U \in \mathsf{Op}(\Omega)$.

Because of this, defining C^k_Ω by setting $C^k_\Omega(U)$ to be all functions $U \to \mathbb{R}$ of class C^k gives a sheaf on Ω.

A restriction of a C^k function $f : U \to \mathbb{R}$ to $V \in \mathsf{Op}(U)$ gives a C^k-function on V. Any function $f : U \to \mathbb{R}$ that has the property of being C^k when restricted to all sets of some open cover $\{U_i\}$ of U, is C^k on the whole of U.

Remark 2.10. The set $\mathcal{F}(U, \mathbb{R})$ is a commutative $\mathbb{R}$-algebra: for $f, g \in \mathcal{F}(U, \mathbb{R})$ and $\lambda \in \mathbb{R}$, one has

$$(f + g)(x) = f(x) + g(x), \ (\lambda f)(x) = \lambda f(x), \ (f \cdot g)(x) = f(x) \cdot g(x).$$

The restrictions along $V \subset U$, $\mathcal{F}(U, \mathbb{R}) \to \mathcal{F}(V, \mathbb{R})$ are $\mathbb{R}$-algebra homomorphisms since we define algebra structures point-by-point.

Definition 2.11. Let X be a topological space and $\mathcal{A}$ a sheaf of functions $\mathcal{A}(U) \subset \mathcal{F}(U, \mathbb{R})$. Then we call $\mathcal{A}$ a *sheaf of* $\mathbb{R}$*-algebras* if each $\mathcal{A}(U)$ is a subalgebra of $\mathcal{F}(U, \mathbb{R})$.

$\mathbb{R}$-spaces

Example 2.12. For any space X, the sheaves $\mathcal{F}_X$ and C_X given by $\mathcal{F}_X(U) := \mathcal{F}(U, \mathbb{R})$ and $C_X(U) := C(U, \mathbb{R})$ are sheaves of $\mathbb{R}$-algebras.

Example 2.13. For $\Omega \subset \mathbb{R}^n$, the sheaves C^k_X of Example 2.9 are sheaves of $\mathbb{R}$-algebras on Ω.

Definition 2.14. Given a map $\varphi : X \to Y$ in Top, $U \in \mathsf{Op}(Y)$ and any $f : U \to \mathbb{R}$, the *pull-back of f along φ*, denoted $\varphi^* f : \varphi^{-1}(U) \to \mathbb{R}$, is the function defined as $(\varphi^* f)(x) := f(\varphi(x))$.

Definition 2.15. A $\mathbb{R}$-*space* is a pair $(X, \mathcal{A})$ of $X \in \mathsf{Top}$ and $\mathcal{A}$ a sheaf of $\mathbb{R}$-algebras on X. A morphism (map) of $\mathbb{R}$-spaces $\varphi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a continuous map $\varphi : X \to Y$ such that for each $U \in \mathsf{Op}(Y)$ and $f \in \mathcal{B}(U)$, the pull-back $\varphi^* f$ belongs to $\mathcal{A}(\varphi^{-1}(U))$.

$\mathbb{R}$-spaces are also called concrete ringed spaces over $\mathbb{R}$ [2].

Remark 2.16. Note that the assignment $\varphi^* : \mathcal{B}(U) \to \mathcal{A}(\varphi^{-1}(U))$, $f \mapsto \varphi^* f$ is automatically an $\mathbb{R}$-algebra homomorphism. Indeed, $\varphi^*(f \cdot g)(x) = (f \cdot g)(\varphi(x)) = f(\varphi(x)) \cdot g(\varphi(x)) = (\varphi^* f \cdot \varphi^* g)(x)$ and so on.

Example 2.17.

1. For a space X, take the sheaf $\mathcal{F}_X$ of all functions to $\mathbb{R}$, $\mathcal{F}_X(U) = \mathcal{F}(U, \mathbb{R})$. Then any continuous map $F : X \to Y$ induces a morphism of $\mathbb{R}$-spaces $(X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ for trivial reasons.
2. For a space X, take the sheaf C_X of continuous functions, $C_X(U) = C(U, \mathbb{R})$. Then any continuous map $F : X \to Y$ induces a morphism of $\mathbb{R}$-spaces $(X, C_X) \to (Y, C_Y)$ since pull-back of any $g \in C(U, \mathbb{R})$ is $g \circ F \in C(f^{-1}(U), \mathbb{R})$ automatically.
3. The pairs (Ω, C^k_Ω) with $\Omega \subset \mathbb{R}^n$ and $C^k_\Omega(U) = C^k(U, \mathbb{R})$ are $\mathbb{R}$-spaces. The identity map id_Ω induces morphisms of $\mathbb{R}$-spaces $(\Omega, C^{k-1}_\Omega) \to (\Omega, C^k_\Omega)$.
4. One can also equip Ω with a sheaf $\mathcal{A}$ where $\mathcal{A}(U)$ are real analytic functions on U. It will be an $\mathbb{R}$-space. The identity id_Ω gives a map $(\Omega, C^\infty_\Omega) \to (\Omega, \mathcal{A})$.

Let $\varphi : X \xrightarrow{\sim} Y$ be a homeomorphism. Then if we consider the $\mathbb{R}$-space structure from Example 2.17, the induced $\mathbb{R}$-space map $(X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ has the property that for any $V \in \mathsf{Op}(Y)$, the map

$$\varphi^* : \mathcal{F}(V, \mathbb{R}) \longrightarrow \mathcal{F}(\varphi^{-1}(V), \mathbb{R}), \quad f \mapsto \varphi^*(f)$$

is an isomorphism of $\mathbb{R}$-algebras, with inverse given by $(\varphi^{-1})^*$. This simply states that functions between isomorphic sets $V \cong \varphi^{-1}(V)$ are the same.

Definition 2.18. An $\mathbb{R}$-space map $\varphi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is an *isomorphism* if $\varphi : X \to Y$ is a homeomorphism and that for any $V \in \mathsf{Op}(Y)$ and $f \in \mathcal{F}(V, \mathbb{R})$, one has

$$f \in \mathcal{B}(V) \iff \varphi^*(f) \in \mathcal{A}(\varphi^{-1}(V)).$$

Note that taking $f = g \circ \varphi^{-1}$ for $g : U \to \mathbb{R}$ defined on $U \in \mathsf{Op}\, X$ yields:

$$g \circ \varphi^{-1} \in \mathcal{B}(\varphi(U)) \iff g \in \mathcal{A}(U).$$

In other words, the sheaves of functions are in bijective correspondence over the corresponding sets.

This definition of isomorphism agrees with notion of composition of $\mathbb{R}$-space maps (see problem in TD).

2.3 Smooth manifolds, defined

If $\mathcal{S}$ is a sheaf on X, then we can restrict it to any open subset $U \subset X$: define $\mathcal{S}|_U$ by $\mathcal{S}|_U(V) = \mathcal{S}(V)$ for $V \in \mathsf{Op}(U)$. It will be again a sheaf.

Example 2.19. If $(X, \mathcal{A})$ is an $\mathbb{R}$-space, then for each $U \in \mathsf{Op}(X)$ the pair $(U, \mathcal{A}|_U)$ is again an $\mathbb{R}$-space. The inclusion $U \hookrightarrow X$ becomes a map $(U, \mathcal{A}|_U) \to (X, \mathcal{A})$.

Definition 2.20. A C^k*-manifold* ($k \in [0, \infty]$) is an $\mathbb{R}$-space $(M, \mathcal{A})$ such that

1. The space M is Hausdorff.
2. For each $x \in M$ there exists $U \in \mathsf{Op}(M)$, $x \in U$, and an $\mathbb{R}$-space isomorphism $(U, \mathcal{A}|_U) \xrightarrow{\sim} (\Omega, C^k_\Omega)$ in the sense of Definition 2.18. Here, $\Omega \in \mathsf{Op}(\mathbb{R}^n)$.

The sheaf $\mathcal{A}$ is then denoted C^k_M, and called the *sheaf of* C^k*-functions* on M. In particular, a C^∞-manifold is called *smooth*. For *any* $U \in \mathsf{Op}\, M$ we shall also write $\mathcal{A}|_U := C^k_U$; the pair (U, C^k_U) is a C^k-manifold.

Remark 2.21. Note that we can replace 2. in the definition above by the condition: M admits an open covering $\{U_i\}_{i\in I}$ such that each $(U_i, \mathcal{A}|_{U_i})$ is isomorphic to some (Ω, C^k_Ω). Another tedious verification permits to give equivalent definitions with $\Omega = \mathbb{R}^n$ or $\mathbb{D}^n$.

Taking $k = 0$ reproduces the definition of topological manifold (Example 2.17.2.)

$\mathbb{S}^n$ is a smooth manifold

Example 2.22. To put a smooth structure on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, we would like to use the map $u : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$, $v \mapsto v/\|v\|$.

1. The map u is continuous (use sequential verification if needed). Moreover $U \subset \mathbb{S}^n$ is open iff so is $u^{-1}(U)$.
2. Define $C^\infty_{\mathbb{S}^n}(U) := \{f : U \to \mathbb{R} \mid u^*(f) : u^{-1}(U) \to \mathbb{R} \text{ is } C^\infty\}$. This set is a $\mathbb{R}$-algebra: since $u^*(\lambda f + \mu g) = \lambda u^*(f) + \mu u^*(g)$, $u^*(fg) = u^*(f)u^*(g)$, we see that pullbacks of sums and products are smooth.
3. If $V \subset U$ then $u^{-1}(V) \subset u^{-1}(U)$. Thus $u^*(f) : u^{-1}(U) \to \mathbb{R}$ being C^∞ implies that $u^*(f) : u^{-1}(V) \to \mathbb{R}$ is C^∞. Hence $C^\infty_{\mathbb{S}^n}$ is a presheaf.
4. If $U = \cup_I U_i$ in $\mathbb{S}^n$ then $u^{-1}(U) = \cup_I u^{-1}(U_i)$ in $\mathbb{R}^{n+1} \setminus 0$. Thus for $f : U \to \mathbb{R}$, if $u^*(f)$ is C^∞ on each $u^{-1}(U_i)$ then $u^*(f)$ is C^∞ on $u^{-1}(U)$ as C^∞ functions form a sheaf on $\mathbb{R}^{n+1} \setminus 0$.
5. **We conclude that $C^\infty_{\mathbb{S}^n}$ is a sheaf and $(\mathbb{S}^n, C^\infty_{\mathbb{S}^n})$ is an $\mathbb{R}$-space.**

6. Let $x \in \mathbb{S}^n$. Assume $x \in U_i^+$ where $U_i^+ = \{(x_0, ..., x_n) \in \mathbb{S}^n \,|\, x_i > 0\}$ is from Example 1.20. We also recall
$$\varphi_i : U_i^+ \xrightarrow{\sim} \mathbb{D}^n, \;\; \varphi_i^+(x_0, ..., x_n) = (x_0, ..., \hat{x}_i, ..., x_n).$$
We want to prove that $\varphi_i : (U_i^+, C^\infty_{U_i^+}) \to (\mathbb{D}^n, C^\infty_{\mathbb{D}^n})$ is an $\mathbb{R}$-space isomorphism.

7. Denote $\mathbb{U}_i^+ := u^{-1}(U_i^+) = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \setminus 0 | x_i > 0\}$. Also denote
$$\Phi_i := \varphi_i \circ u : \mathbb{U}_i^+ \to \mathbb{D}^n, \;\; \Phi(x_0, ..., x_n) = \frac{1}{\|x\|}(x_0, ..., \hat{x}_i, ..., x_n).$$

8. The map Φ_i is smooth. If $V \in \mathsf{Op}(\mathbb{D}^n)$ and $f : V \to \mathbb{R}$ is C^∞, then so is $\Phi_i^*(f)$.

9. Let $V \in \mathsf{Op}(\mathbb{D}^n)$. Take $f : V \to \mathbb{R}$ of class C^∞, check if $\varphi_i^*(f) \in C^\infty_{U_i^+}(\varphi_i^{-1}(V))$. For this, note that $u^*(\varphi_i^*(f)) = f \circ \varphi_i \circ u = f \circ \Phi_i$. This is C^∞ by 8.

10. Let $V \in \mathsf{Op}(\mathbb{R}^n)$. Take $f : V \to \mathbb{R}$ and assume that $\varphi_i^*(f) \in C^\infty_{U_i^+}(\varphi_i^{-1}(V))$. This means that, by calculus above, the function $f \circ \Phi_i : \Phi_i^{-1}(V) \to \mathbb{R}$ is C^∞. Note that
$$\eta_i^+(y_1, ..., y_n) = (y_1, ..., y_{i-1}, \sqrt{1 - \sum_{j=1}^n y_j^2}, y_i, ..., y_n)$$
can be viewed as a smooth function from $\mathbb{D}^n$ to $\mathbb{U}_i^+$. We have $\Phi_i \circ \eta_i^+ = \mathsf{id}_{\mathbb{D}^n}$ and so $f = f \circ (\Phi_i \circ \eta_+^i) = (f \circ \Phi_i) \circ \eta_+^i$ is smooth as well.

11. **This proves that $\varphi_i : (U_i^+, C^\infty_{U_i^+}) \to (\mathbb{D}^n, C^\infty_{\mathbb{D}^n})$ is an $\mathbb{R}$-space isomorphism.**

12. The same arguments can be done for U_i^- instead of U_i^+. We conclude since $\mathbb{S}^n = \big(\cup_{i=0}^n U_i^+\big) \cup \big(\cup_{i=0}^n U_i^-\big)$.

$\mathbb{RP}^n$ is a smooth manifold

Example 2.23. Recall $q : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$, $v \mapsto \mathcal{L} = \mathsf{Span}(v)$. We constructed an open chart using $C = e + H$ where $H = \{e\}^\perp$. We now define $C^k_{\mathbb{RP}^n}$ by setting
$$C^k_{\mathbb{RP}^n}(U) := \{f : U \to \mathbb{R} \,|\, q^*(f) = f \circ q : q^{-1}(U) \to \mathbb{R} \text{ is } C^k\}.$$
This set is a $\mathbb{R}$-algebra: since $q^*(\lambda f + \mu g) = \lambda q^*(f) + \mu q^*(g)$, $q^*(fg) = q^*(f)q^*(g)$, we see that pullbacks of sums and products are smooth.

1. If $V \subset U$ then $q^{-1}(V) \subset q^{-1}(U)$. Thus $q^*(f) : q^{-1}(U) \to \mathbb{R}$ being C^k implies that $q^*(f) : q^{-1}(V) \to \mathbb{R}$ is C^k. Hence $C^k_{\mathbb{RP}^n}$ is a presheaf.

2. If $U = \cup_I U_i$ in $\mathbb{RP}^n$ then $q^{-1}(U) = \cup_I q^{-1}(U_i)$ in $\mathbb{R}^{n+1} \setminus 0$. Thus for $f : U \to \mathbb{R}$, if $q^*(f)$ is C^k on each $q^{-1}(U_i)$ then $q^*(f)$ is C^k on $q^{-1}(U)$ as C^k functions form a sheaf on $\mathbb{R}^{n+1} \setminus 0$.

3. **We conclude that $C^k_{\mathbb{RP}^n}$ is a sheaf and $(\mathbb{RP}^n, C^k_{\mathbb{RP}^n})$ is an $\mathbb{R}$-space.**

4. Recall map $p_C : \mathbb{R}^{n+1} \setminus H \to C$, $ae + h \mapsto e + h/a$ used to construct homeomorphism $\bar{p}_C : \mathbb{RP}^n \setminus q(H) \xrightarrow{\sim} C$ Choose $e = e_0$ of the standard basis and identify $e_0 + H_0$ with $H_0 = \{(x_1, ..., x_n)\}$ using $e + h \mapsto h$. Then the map p_C gives rise to map
$$p_0 : U_0 := \mathbb{R}^{n+1} \setminus H_0 \longrightarrow \mathbb{R}^n, \;\; (x_0, ..., x_n) \mapsto (x_1/x_0, ..., x_n/x_0)$$
which in turn gives a homeomorphism $\bar{p}_0 : \bar{U}_0 \xrightarrow{\sim} \mathbb{R}^n$.

5. Moreover, the map p_0 is smooth. Thus is $V \in \mathsf{Op}(\mathbb{R}^n)$ and $f : V \to \mathbb{R}$ is C^k, then so is $p_0^*(f)$.

6. Let $V \in \mathsf{Op}(\mathbb{R}^n)$. Take $f : V \to \mathbb{R}$ of class C^k, check if $\bar{p}_0^*(f) \in C^k_{\bar{U}_0}(\bar{p}_0^{-1}(V))$. For this, note that $q^*(\bar{p}_0^*(f)) = f \circ \bar{p}_0 \circ q = f \circ p_0$. This is C^k by 5.

7. Let $V \in \mathsf{Op}(\mathbb{R}^n)$. Take $f : V \to \mathbb{R}$ and assume that $\bar{p}_0^*(f) \in C^k_{\bar{U}_0}(\bar{p}_0^{-1}(V))$. This means that, by calculus above, the function $f \circ p_0 : p_0^{-1}(V) \to \mathbb{R}$ is C^k. In coordinates, $(f \circ p_0)(x_0, ..., x_1) = f(x_1/x_0, ..., x_n/x_0)$. We see that $f = f \circ p_0 \circ i_0$ where $i_0(x_1, .., x_n) = (1, x_1, ...x_n)$ a smooth map with domain $\mathbb{R}^n$. Associativity of composition gives that f is smooth.

8. **We conclude that $\bar{p}_0 : (\bar{U}_0, C^k_{U_0}) \xrightarrow{\sim} (\mathbb{R}^n, C^k_{\mathbb{R}^n})$ is an $\mathbb{R}$-space isomorphism.**

9. We can similarly define $U_i = \{(x_0, ..., x_n) | x_i \neq 0\}$, $p_i : U_i \to \mathbb{R}^n$ and prove that $\bar{p}_i : (\bar{U}_i, C^k_{U_i}) \xrightarrow{\sim} (\mathbb{R}^n, C^k_{\mathbb{R}^n})$ is an $\mathbb{R}$-space isomorphism. We finally observe that $\mathbb{RP}^n = \bar{U}_0 \cup ... \cup \bar{U}_n$.

10. Of course, this is valid for any k, so in particular $\mathbb{RP}^n$ is a smooth manifold. The Hausdorff condition is easy to verify using charts.

Exercise 2.24. *Denote by q the defining projection $\mathbb{R}^2 \to \mathbb{T}$. For $U \in \mathsf{Op}(\mathbb{T})$, define $C^\infty_{\mathbb{T}}(U) := \{f : U \to \mathbb{R} \,|\, q^*(f) : q^{-1}(U) \to \mathbb{R}$ is $C^\infty\}$. Prove that $(\mathbb{T}, C^\infty_{\mathbb{T}})$ is a C^∞-manifold.*

2.4 Revisiting atlases

Definition 2.25. For a C^k-manifold (M, C^k), the $\mathbb{R}$-space isomorphism $\varphi : (U, C^\infty_U) \xrightarrow{\sim} (\Omega, C^k_\Omega)$ for $U \in \mathsf{Op}(M)$ is called an *open chart*. It is centred at $p \in U$ if $\varphi(p) = 0$. The number n in $\Omega \subset \mathbb{R}^n$ is called the *dimension* of M.

Remark 2.26. A topological manifold M that can be made into a C^1-manifold can also be made into a C^∞-manifold. So we will often put $k = \infty$.

A C^k-manifold (M, C^k_M) has no compatibility issues. Let $\varphi : (U, C^k_U) \xrightarrow{\sim} (\Omega, C^k_\Omega)$, $\psi : (V, C^k_V) \xrightarrow{\sim} (\Theta, C^k_\Theta)$ be two coordinate charts. Let $W \subset U \cap V$ open. Then $C^k_U(W) = C^k_M(W) = C^k_V(W)$ by definition. Spelling out in detail the definition, it means that

$$\{f : W \to \mathbb{R} \,|\, f \circ \varphi^{-1} \in C^k_\Omega(\varphi(W))\} = \{f : W \to \mathbb{R} \,|\, f \circ \psi^{-1} \in C^k_\Theta(\psi(W))\}$$

In particular, take a function $p_i : \psi(W) \to \mathbb{R}$, $p_i(x_1, ..., x_n) = x_i$. Evidently $p_i \in C^k_\Theta(\psi(W))$. Consider then $f = p_i \circ \psi : W \to \mathbb{R}$. The RHS of the above equality implies that $f \in C^k_M(W)$, and so using the LHS we see that $p_i \circ \psi \circ \varphi^{-1} : \varphi(W) \to \mathbb{R}$ is C^k.

We conclude that $\psi \circ \varphi^{-1} : \varphi(W) \to \psi(W) \subset \mathbb{R}^n$ is of class C^k. Can interchange φ and ψ as their role is symmetric. Atlas returns and is subsumed.

Formalising passage from atlas to manifold

Lemma 2.27. *Let U be a Hausdorff topological space and $\varphi : U \xrightarrow{\sim} \Omega$ a homeomorphism onto $\Omega \subset \mathbb{R}^n$. Define C^k_U to be the presheaf*

$$C^k_U(V) := \{f : V \to \mathbb{R} \mid f \circ \varphi^{-1} : \varphi(V) \to \mathbb{R} \text{ is } C^k\}.$$

Then (U, C^k_U) is a C^k-manifold and $\varphi : (U, C^k_U) \to (\Omega, C^k_\Omega)$ is an $\mathbb{R}$-space isomorphism. The statement works for any k.

Proof. Using the identities $(\lambda f + \mu g) \circ \varphi^{-1} = \lambda(f \circ \varphi^{-1}) + \mu(g \circ \varphi^{-1})$, $(f \cdot g) \circ \varphi^{-1} = (f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1})$ we see that $C^k_U(V)$ is stable by linear sums and products.

If $V_1 \subset V_2$ then $\varphi(V_1) \subset \varphi(V_2)$ so if $f : V_2 \to \mathbb{R}$ and $f \circ \varphi^{-1}$ is C^k on $\varphi(V_2)$ then the same is true on $\varphi(V_1)$.

Let $V = \cup_i V_i$. Then, $\varphi(V) = \cup_i \varphi(V_i)$. If $f : V \to \mathbb{R}$ is such that $f \circ \varphi^{-1} : \varphi(V_i) \to \mathbb{R}$ is C^k for each i, then the same is true for $f \circ \varphi^{-1} : \varphi(V) \to \mathbb{R}$ since C^k_Ω is a sheaf. Thus $f \in C^k_U(V)$ as required.

Finally note that for $W \in \mathsf{Op}\,\Omega$ and $f : W \to \mathbb{R}$, $f \in C^k_\Omega(W)$ implies that $\varphi^* f \in C^k_U(\varphi^{-1}(W))$ and vice versa, since $(\varphi^* f) \circ \varphi^{-1} = f$. $\square$

Let M be a Hausdorff topological space and $\mathcal{A} = \{(U_i, \varphi_i : U_i \xrightarrow{\sim} \Omega_i)\}_{i \in I}$ be a C^k-atlas as per Definition 2.3.

Let us define the presheaf C^k_M by declaring $C^k_M(V)$ to consist of all functions $f : V \to \mathbb{R}$ satisfying: for each point $x \in V$ there is a chart (U_i, φ_i) containing x such that $f \circ \varphi_i^{-1} : \varphi_i(V \cap U_i) \to \mathbb{R}$ is C^k.

Proposition 2.28. *Then the pair (M, C^k_M) is a C^k-manifold.*

Proof. Tedious. Some elements:

1. Let V be an open subset of some U_i. Then we say that $f : V \to \mathbb{R}$ is φ_i-smooth if $f \circ \varphi_i^{-1} : \varphi_i(V) \to \mathbb{R}$ is C^k. Thanks to the previous lemma φ_i-smooth functions form a sheaf of algebras on U_i.

2. The definition of C^k_M can be rephrased as follows. For a $W \in \mathsf{Op}\,M$ and $f : W \to \mathbb{R}$, we say that $f \in C^k_M(W)$ iff there exists a subcollection of atlas charts $\{U_j\}_{j \in J}$ such that $W \subset \cup_J U_J$ and $f|_{W \cap U_j}$ is φ_j-smooth for each $j \in J$.

3. Let $W \in \mathsf{Op}\,M$, and $\{U_j\}_{j \in J}$, $\{U_\alpha\}_{\alpha \in A}$ be two collections of charts so that $\cup_J U_j \supset W \subset \cup_A U_\alpha$. Let $f : W \to \mathbb{R}$ be such that $f|_{W \cap U_j}$ is φ_j-smooth for each $j \in J$. Then for each j, α, $f|_{W \cap U_j \cap U_\alpha}$ is φ_j-smooth. Due to the compatibility of the atlas it is the same as being φ_α-smooth.

   Since φ_α-smooth functions form a sheaf on U_α, we conclude that $f|_{W \cap U_\alpha}$ is φ_α-smooth for each $\alpha \in A$. As a result, the definition of $C^k(W)$ is independent of the cover.

4. We can in fact always work with the total cover $\{U_i\}_{i \in I}$. If $U_i \cap W = \emptyset$, then the restriction of $f : W \to \mathbb{R}$ to $\emptyset$ is the "do-nothing function" that is automatically assumed to be φ_i-smooth.

5. The remaining verifications are done chart-by-chart. For example, if $V_1 \subset V_2$ in M then $V_1 \cap U_i \subset V_2 \cap U_i$ in U_i. So if $f : V_2 \to \mathbb{R}$ is φ_i-smooth on $U_i \cap V_2$, then it is also φ_i-smooth on $U_i \cap V_1$. This gives the presheaf property.

6. If $V = \cup_J V_j$ and $f : V \to \mathbb{R}$ has the property that $f|_{V_j \cap U_i}$ is φ_i-smooth for all $(i, j) \in I \times J$, then use $V \cap U_i = \cup_J (V_j \cap U_i)$ to conclude that $f|_{V \cap U_i}$ is φ_i-smooth for each i. This gives the sheaf property.

7. Finally to verify that $C^k_M(W)$ is a sheaf of subalgebras we check that it is the case on each U_i since algebra operations are defined pointwise.

8. We have shown that (M, C^k_M) is an $\mathbb{R}$-space. If we restrict to $U_i \cong \Omega_i$, then for $V \in \mathsf{Op}\,\Omega_i$ and $f : V \to \mathbb{R}$,
$$f \in C^k_{\Omega_i}(V) \iff \varphi^*(f) \in C^k_{U_i}(\varphi^{-1}(V))$$
practically by definition. □

Remark 2.29. (do we really need $\mathbb{R}$-spaces?) Our considerations show that the fact that $C^k_M(U)$ is a $\mathbb{R}$-subalgebra is mostly formal. In fact, one can give the following, simpler definition of a C^k-manifold.

It is a pair $(M, \mathcal{O})$ with M Hausdorff and $\mathcal{O}$ a sheaf of $\mathbb{R}$-valued functions on M such that for each x there exists an open U containing x, a homeomorphism $\varphi : U \xrightarrow{\sim} \Omega$, such that for each $V \in \mathsf{Op}\,\Omega$ and $f : V \to \mathbb{R}$, one has
$$f \in C^k_\Omega(V) \iff \varphi^*(f) \in \mathcal{O}(\varphi^{-1}V).$$
This results in $(U, \mathcal{O}|_U)$ being an $\mathbb{R}$-space and we then cover M by all possible U to see that $\mathcal{O} \equiv C^k_M$ is a sheaf of subalgebras on M.

The fact that (M, C^k_M) is an $\mathbb{R}$-space is used all the time in this course and conforms to the literature [2] but you are free to use this weaker definition.

3 Smooth maps and differentials

3.1 Bump functions and partitions of unity

Definition 3.1. Let (M, C^k_M) be a C^k manifold. The $\mathbb{R}$-algebra of C^k functions on M is defined as $C^k(M) := C^k_M(M)$.

We can of course restrict smooth functions: $C^k(M) \to C^k_M(U)$ for $U \in \mathsf{Op}(M)$. However, we can also "go back" in certain cases, thanks to some particular smooth functions that exist.

Definition 3.2. Let $f \in C^k(M)$. Define the *support of* f as
$$\operatorname{supp} f := \overline{\{x \in M \mid f(x) \neq 0\}}.$$

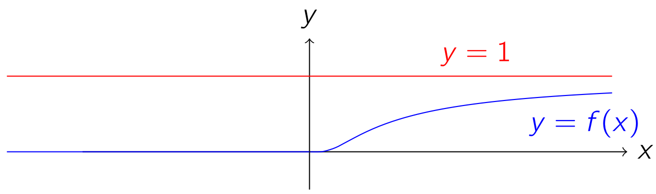
Lemma 3.3. *There exists a smooth function* $H : \mathbb{R}^n \to [0, 1]$ *such that*

1. $H(x) = 1$ *for* $\|x\| \leq 1$,
2. $H(x) \in]0, 1]$ *for* $1 \leq \|x\| < 2$
3. $\operatorname{supp} H$ *is contained in* $\{\|x\| \leq 2\}$*: in other words* $H(x) = 0$ *for* $\|x\| \geq 2$.

Proof. First, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(t) = 0$ if $t \leq 0$ and $f(t) = \exp(-1/t)$ otherwise. For $t > 0$, one proves by induction that

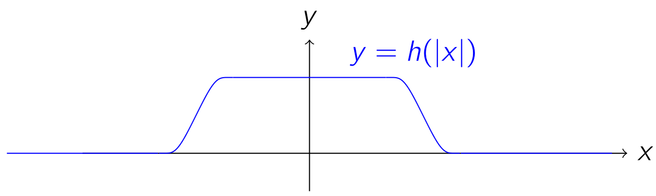
$$f^{(n)}(t) = \frac{P_n(t)}{t^{2n}} \exp(-1/t)$$

where P_n is some polynomial. Because of this $\lim_{t \to 0^+} f^{(n)}(t) = 0$ and so f is smooth.



The rest of the proof consists in observing that $h : \mathbb{R} \to \mathbb{R}$, $h(t) = \frac{f(2-t)}{f(2-t)+f(t-1)}$ satisfies the following conditions:

1. it is well-defined, since for all t either $2 - t$ or $t - 1$ is positive.
2. when $t \leq 1$, $f(t-1) = 0$ and so $h(t) \equiv 1$.
3. $0 < h(t) < 1$ when $t \in]1, 2[$ since the function f takes values in $[0, 1[$.
4. $h(t) \equiv 0$ for $t \geq 2$ due to its numerator turning to zero.
5. h is smooth on $\mathbb{R}$.



We conclude our proof by defining $H(x) := h(\|x\|)$. This function is smooth everywhere, including at 0. $\square$

Remark 3.4. The functions like f or h in the proof above are smooth but not real-analytic. Indeed, the Taylor series of f at $t_0 = 0$ is $0 \neq \exp(-1/t)$. The results of this subsection are specific to smooth manifold calculus.

Corollary 3.5. *Let M be a C^k-manifold of dimension n. Then for each $p \in M$ there exists an open chart V_p, a $W_p \in \mathsf{Op}(V_p)$ such that $\overline{W_p} \subset V_p$ and a "bump" function $f \in C^k(M)$ such that*

1. $0 \leq f(x) \leq 1$ *for all* $x \in M$,
2. $f|_{\overline{W_p}} \equiv 1$,
3. $\mathsf{supp}\, f = \overline{\{x \in M \mid f(x) \neq 0\}} \subset V_p$.

Proof. Recall the notation $B(0, \varepsilon) := \{x \in \mathbb{R}^n \mid \|x\| < \varepsilon\}$. One can always find a C^k-chart $\varphi : U_p \xrightarrow{\sim} B(0, 3)$ centred at p. Put $V_p := \varphi^{-1}(B(0, 2\frac{1}{2}))$ and $W_p := \varphi^{-1}(B(0, 1))$.

Define f by setting

$$f(x) = \begin{cases} H \circ \varphi(x) & \text{if } x \in U_p \\ 0 & \text{if } x \in M \setminus \overline{V_p} \end{cases}$$

with function H as in Lemma 3.3. Since H is zero outside $B(0, 2)$, the expression is well-defined on $U_p \setminus \overline{V_p}$.

Due to the properties of H, it remains to prove that $f : M \to \mathbb{R}$ is in $C^k(M)$. However $M = (M \setminus \overline{V_p}) \cup U_p$ and the restrictions of f to $M \setminus \overline{V_p}$ and U_p are C^k by definition. Since C^k functions form a sheaf, we conclude. $\square$

Remark 3.6. Since an open subset of a manifold is a manifold, we can state the following variation of the previous corollary: for any $p \in M$ and any open U containing p there exists open W_p, $p \in \overline{W_p} \subset U$, and a function f taking values in $[0, 1]$ that is $\equiv 1$ on $\overline{W_p}$ and has $\mathsf{supp}\, f \subset U$.

Still, how did we use that M is Hausdorff?

In fact, if we inspect closely the preceding proof, the set $\overline{V_p} = \varphi^{-1}(\overline{B(0, 2\frac{1}{2})})$ is guaranteed to be closed *as a subset of* U_p. Why is it a closed subset of the whole of M?

For this it is enough to observe that $\overline{V_p}$ is compact in M: it the image of a compact set, the closed unit ball, and compacts in U_p remain compacts in M.

Hausdorff property then implies that compact sets are closed, so $\overline{V_p}$ is closed in M. If S is any closed subset containing V_p then $S \cap U_p$ is a closed set of U_p and thus has to contain $\overline{V_p}$. The closure notation is thus sensible.

How do we prove that compact in Hausdorff is closed? Let K be compact for induced topology in a Hausdorff space X. Let $x \in X \setminus K$. For each $y \in Y$ we can find U_y, V_y, $x \in U_y, y \in V_y$, $U_y \cap V_y = \emptyset$. Since K is compact if follows tha there exists a finite number of $y_1, ..., y_n$ such that $K \subset V_{y_1} \cup ... \cup V_{y_n}$.

Putting $U := U_{y_1} \cap ... \cap U_{y_n}$ produces an open set containing x and not intersecting any of V_{y_i}, hence not intersecting K. This means that K is closed.

Partitions of unity

(This material is needed much later)

First, let us discuss the compact case.

Proposition 3.7. *Let (M, C^k_M) be a C^k-manifold, and $\{U_i\}_{i \in I}$ be an open cover of M. Assume that M is compact.*

Then there exists a set of functions $\{p_i\}_{i \in I}$, $p_i \in C^k(M)$ such that

1. $p_i(x) \in [0, 1]$ *for all* $x \in M$ *and* $i \in I$,

2. $\operatorname{supp} p_i \subset U_i$,

3. *the set* $\{\operatorname{supp} p_i\}_{i\in I}$ *is locally finite: for each* $x \in M$ *there is an open* $U \ni x$ *that intersects only with a finite number of* $\operatorname{supp} p_i$,

4. $\sum_{i\in I} p_i(x) = 1$ *for all* $x \in M$.

Definition 3.8. Given a manifold (M, C^k_M) and an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M, the set of functions $\{p_i\}_{i\in I}$ with the properties $1. - 4.$ above is called a *partition of unity subordinate to* $\mathcal{U}$.

Proof.

1. Due to compactness we can choose a finite subcover $U_{i_1} \cup ... \cup U_{i_m} = M$ and put $p_i(x) = 0$ for $i \neq i_1, ..., i_m$. We hence assume that $I = \{1, ..., m\}$.

2. For each point $x \in M$, we recall the existence of (V_x, W_x, f) as in Corollary 3.5. Each $(U_i, C^k_{U_i})$ is a manifold, so if $x \in U_i$ we can find (V_x, W_x, f) with $V_x \subset U_i$.

3. Do that for each i, conclude: there exists some (potentially infinite) index set A and points $\{x_\alpha\}_{\alpha\in A}$ such that the associated $(V_{x_\alpha}, W_{x_\alpha}, f_\alpha)$ satisfy

   - for each α, the set V_{x_α} (and thus W_{x_α}) is contained in some U_i,
   - the sets W_{x_α} (and thus V_{x_α}) cover M.

   One says in such situation that the coverings $\{V_{x_\alpha}\}_A$ and $\{W_{x_\alpha}\}_A$ *refine* the covering $\{U_i\}_I$.

4. Use compactness of M once again to choose a finite set of indices $\alpha_1, ..., \alpha_l$ so that $\{W_{x_\alpha}\}_{\alpha_1,...\alpha_l}$ is a cover. Simplify the notation by assuming $\alpha \in \{1, ..., l\}$ and declaring $V_\alpha = V_{x_\alpha}$, $W_\alpha = W_{x_\alpha}$.

5. Define $f(x) := \sum_\alpha f_\alpha(x)$. This is a function in $C^k(M)$ that is strictly positive since W_α form a cover and each $f_\alpha \equiv 1$ on W_α. Now define $q_\alpha(x) := f_\alpha(x)/f$. This function is C^k on M (division by positive C^k functions is defined locally and gives C^k functions).

6. By construction $\{q_\alpha\}_\alpha$ is a partition of unity subordinate to $\{V_\alpha\}$.

7. We conclude by setting $p_i = \sum_{\alpha\in A_i} q_\alpha$, where $A_i = \{\alpha | V_\alpha \subset U_i\}$. □

What about non-compact?

If M is non-compact, then the problem in the construction above is apparent: too many open sets covering the same neighbourhood. Because of this we cannot sum the functions f_α.

It turns out that one needs to impose the following equivalent axioms on M:

1. the topology on M is metrisable: there exists a distance function d such that the metric space (M, d) gives the topology on M,

2. each connected component of the space M admits a countable basis for its topology,

3. the space M is paracompact: any open cover $\{U_i\}$ admits a locally finite refinement.

Most non-compact manifolds satisfy this: $\mathbb{R}^n$, opens in $\mathbb{R}^n$, things glued from countable amount of charts. We will not delve into these details and cite our references [3, 2, 5] for detailed proofs. **From now on, we assume that all manifolds M admit partitions of unity subordinate to any cover.**

Corollary 3.9. *Let M be a C^k-manifold. Then for any A closed subset of M and any $U \in \mathsf{Op}(M)$, $A \subset U$, there exists $b \in C^k(M)$ such that $b|_A \equiv 1$ and $\operatorname{supp} b \subset U$.*

Proof. Let (p_0, p_1) be a partition of unity subordinate to the cover $(U, M \setminus A)$. For $x \in A$, we have that $1 = p_0(x) + p_1(x) = p_0(x)$ since $\operatorname{supp} p_1 \subset M \setminus A$. Thus $b = p_0$ works. ☐

Definition 3.10. Let M be a C^k-manifold. Let S be any subset of M. Define

$$C^k_M(S) := \{f : S \to M \mid \exists U \in \mathsf{Op}(M), S \subset U, g \in C^k_M(U) : \ f = g|_S\}.$$

Corollary 3.11. *Let S be closed. Then any $f \in C^k_M(S)$ and any $U \in \mathsf{Op}\, M$, $S \subset U$, there exists a function $\tilde{f} \in C^k(M)$ such that $\tilde{f}|_S = f$ and $\operatorname{supp} \tilde{f} \subset U$.*

Proof. By definition there exists $W \in \mathsf{Op}(M)$ and $g \in C^k_M(W)$ that extends f. Let U be an open containing S. Let $V = U \cap W$ and $h = g|_V$. Then we choose a function $b \in C^k(M)$ as in Corollary 3.9 for $S \subset V$. Taking

$$\tilde{f}(x) = \begin{cases} h(x)b(x) & \text{if } x \in V \\ 0 & \text{if } x \in M \setminus \operatorname{supp} b \end{cases}$$

Since $M = (M \setminus \operatorname{supp} b) \cup V$ and C^k_M is a sheaf, $\tilde{f} \in C^k(M)$. ☐

3.2 Smooth maps

Definition 3.12. Let (M, C^k_M), (N, C^k_N) be two C^k-manifolds (not necessarily of the same dimension). A *C^k-map* from M to N is simply an $\mathbb{R}$-space map $f : (M, C^k_M) \to (N, C^k_N)$. It is called a *diffeomorphism* if it admits an inverse that is also a C^k-map.

Example 3.13.
1. A map between opens of Euclidean spaces is C^k iff it is C^k in the ordinary sense.
2. Thus the multiplication map $\mathsf{GL}_n(\mathbb{R}) \times \mathsf{GL}_n(\mathbb{R}) \to \mathsf{GL}_n(\mathbb{R})$ is smooth.
3. The maps $q : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ and $u : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$ are C^∞ (by construction).
4. The antipodal map $a : \mathbb{S}^n \to \mathbb{RP}^n$, $x \mapsto \mathsf{Span}(x)$ fits into the following commutative diagram

$$\begin{tikzcd} & \mathbb{R}^{n+1} \setminus 0 \arrow[dl, "u"'] \arrow[dr, "q"] & \\ \mathbb{S}^n \arrow[rr, "a"'] & & \mathbb{RP}^n. \end{tikzcd}$$

Since $f \in C^\infty_{\mathbb{RP}^n}(U) \Leftrightarrow q^*(f) \in C^\infty_{\mathbb{R}^{n+1}\setminus 0}(q^{-1}(U))$ and $q^* = u^* \circ a^*$ we have that $u^*(a^*(f))$ is smooth on an open of $\mathbb{R}^{n+1} \setminus 0$ and hence $a^*(f) \in C^\infty_{\mathbb{S}^n}(a^{-1}(U))$.

5. The inclusion map $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth. This follows from the fact that $u \circ i$ is smooth and pullback arguments similar to 4.

6. Define $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^3$ by the following rule:

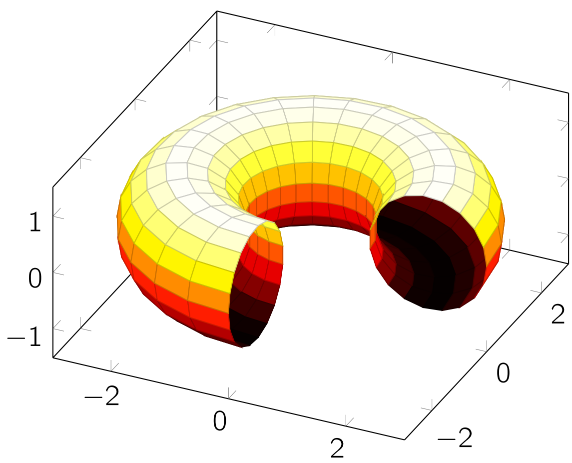
$$\tilde{F}(x, y) = ((2 + \cos 2\pi x) \cos 2\pi y, (2 + \cos 2\pi x) \sin 2\pi y, \sin 2\pi x).$$

   This map is written using smooth functions, and is thus smooth (in manifold or ordinary sense). Note that $\tilde{F}(x + n, y + m) = \tilde{F}(x, y)$ And thus we can induce a continuous map $F : \mathbb{T} \to \mathbb{R}^3$. Note that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dl, "q"'] \arrow[dr, "\tilde{F}"] & \\ \mathbb{T} \arrow[rr, "F"'] & & \mathbb{R}^3 \end{tikzcd}$$

   Because of this, if $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth then $f \circ F : \mathbb{T} \to \mathbb{R}$ is smooth on $\mathbb{T}$: indeed, $q^*(f \circ F) = f \circ F \circ q = f \circ \tilde{F}$, and the latter is smooth due to the properties of $\tilde{F}$.

   One can check that the image F (or of $\tilde{F}$) is the set $\{(y - 2)^2 + z^2 = 1\}$ rotated around the z-axis.



Characterisation of smooth maps

Proposition 3.14. *Let M, N be two C^k-manifolds. A map $f : M \to N$ is C^k if one of the following holds:*

1. *(Definitional) f is an $\mathbb{R}$-space map $(M, C^k_M) \to (N, C^k_N)$.*

2. *(Charts) for each open chart $\varphi : U \xrightarrow{\sim} \Omega$ of M and each open chart $\psi : V \xrightarrow{\sim} \Theta$ of N the map ($W = f^{-1}(V) \cap U$)*

$$\tilde{f} := \psi \circ f \circ \varphi^{-1} : \varphi(W) \to \Theta$$

   is smooth.

3. *(Charts II) for each $x \in M$ there exists an open chart $\varphi : U \xrightarrow{\sim} \Omega$ of M containing x and an open chart $\psi : V \xrightarrow{\sim} \Theta$ of N containing $f(x)$ such that the map ($W = f^{-1}(V) \cap U$)*

$$\tilde{f} := \psi \circ f \circ \varphi^{-1} : \varphi(W) \to \Theta$$

is smooth.

4. *(Global functions only) for each $g \in C^k(N)$ one has $f^*(g) \in C^k(M)$.*

On commutative diagrams

A comment before we start proving. I keep using commutative diagrams. They are a very nice way of book-keeping various operations. We can consider commutative squares and triangles:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[rr, "h"'] & & C, \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "p"] \arrow[d, "r"'] & B' \arrow[d, "s"] \\ C' \arrow[r, "q"'] & D', \end{tikzcd}$$

where $A, B, C; A', B', C', D'$ some mathematical objects (sets, vector spaces, topological spaces etc) and $f, g, h; p, q, r, s$ some maps of those objects (functions, linear maps, continuous maps). The drawn diagrams state that $h \circ f = g$ and $q \circ r = s \circ p$.

Besides a visual presentation, these diagrams allow to do some computations. Indeed, we can play constructor and make more complex diagrams out of simple ones.

Example:

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ C \arrow[u, "g"] \arrow[r, "i"] \arrow[d, "j"'] & D \arrow[u, "h"'] \arrow[d, "k"] \\ E \arrow[r, "l"'] & F, \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B' \arrow[d, "\gamma"] \\ C' \arrow[r, "\delta"] \arrow[d, "\epsilon"'] & D' \arrow[d, "\zeta"] \\ E' \arrow[r, "\eta"'] & F'. \end{tikzcd}$$

The left one tells us two identities: $f \circ g = h \circ i$ and $l \circ j = k \circ i$. The right one however implies that the following square is also commutative:

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "\epsilon \circ \beta"'] & B' \arrow[d, "\zeta \circ \gamma"] \\ E' \arrow[r, "\eta"'] & F'. \end{tikzcd}$$

Indeed,

$$\eta \circ (\epsilon \circ \beta) = (\eta \circ \epsilon) \circ \beta = (\zeta \circ \delta) \circ \beta = \zeta \circ (\delta \circ \beta) = \zeta \circ (\gamma \circ \alpha) = (\zeta \circ \gamma) \circ \alpha.$$

One other nice property is the possibility to reverse invertible maps. For example, assume given a commutaative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"'] & D \end{tikzcd}$$

such that there are f^{-1}, i^{-1}. Then the following diagram is also commutative:

$$\begin{tikzcd} A \arrow[d, "g"'] & B \arrow[l, "f^{-1}"'] \arrow[d, "h"] \\ C & D. \arrow[l, "i^{-1}"] \end{tikzcd}$$

This happens because

$$g \circ f^{-1} = i^{-1} \circ i \circ g \circ f^{-1} = i^{-1} \circ h \circ f \circ f^{-1} = i^{-1} \circ h.$$

Proof.

$1 \Rightarrow 2$: Take $\varphi : U \overset{\sim}{\to} \Omega$, $\psi : V \overset{\sim}{\to} \Theta$, $W = f^{-1}(V) \cap U$. The following diagram commutes ($\tilde{f} := \psi \circ f \circ \varphi^{-1}$):

$$\begin{tikzcd} W \arrow[d, "f|_W"'] & \varphi(W) \arrow[l, "\varphi^{-1}"'] \arrow[d, "\tilde{f}"] \\ V \arrow[r, "\psi"'] & \Theta. \end{tikzcd}$$

For this reason, if we take $h \in C^k(\Theta)$, its pullback $\tilde{f}^*(h)$ coincides with

$$\tilde{f}^*(h) = h \circ \tilde{f} = ((h \circ \psi) \circ f) \circ \varphi^{-1};$$

we can put brackets as we want due to associativity.

Since (V, ψ) is a chart we have $h \circ \psi = \psi^*(h) \in C^k_N(V)$, then by $\mathbb{R}$-space map property $f^*(\psi^*(h)) = h \circ \psi \circ f$ belongs to $C^k_M(W)$, and so its pullback by φ^{-1} belongs to $C^k(\varphi(W))$. Thus $\tilde{f}^*(h) \in C^k(W)$.

Taking $h = p_j$ where $p_j(y_1, ..., y_n) = y_j$ is the j-th coordinate projection means that $\tilde{f}_j$ is C^k. We conclude that $\tilde{f}$ is C^k.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 4$: Let $g \in C^k(N)$ and $x \in M$. Choose φ, ψ as in 3. so that the following diagram commutes:

$$\begin{tikzcd} W \arrow[d, "f|_W"'] & \varphi(W) \arrow[l, "\varphi^{-1}"'] \arrow[d, "\tilde{f}"] \\ V & \Theta \arrow[l, "\psi^{-1}"] \end{tikzcd}$$

$(x \in W)$. Consider

$$f|_W^*(g|_V) = f^*(g)|_W : W \to \mathbb{R}.$$

We want to show that $f^*(g) \circ \varphi^{-1} : \varphi(W) \to \mathbb{R}$ is C^k. This will suffice since we can cover M by opens like W and use sheaf condition. Note that

$$g \circ f \circ \varphi^{-1} = (g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) = (g \circ \psi^{-1}) \circ \tilde{f}.$$

Here $(g \circ \psi^{-1})$ is C^k on Θ. We are thus pre-composing a (restriction of) a C^k function by a C^k-function, that will remain C^k.

$4 \Rightarrow 1$: Let $g \in C^k_N(V)$. Following similar reasoning as in Corollary 3.5 and Remark 3.6, for each point $y \in V$ there exists an open neighbourhood W_y of y such that $\overline{W_y} \subset V$, and a function $b \in C^k(N)$ that is unity on $\overline{W_y}$ and has $\operatorname{supp} b \subset V$.

Let $\tilde{g}$ be a function on M defined as

$$\tilde{g}(x) = \begin{cases} g(x)b(x) & \text{if } x \in V \\ 0 & \text{if } x \in M \setminus \operatorname{supp} b \end{cases}$$

since $\operatorname{supp} b \subset V$, one has $M = (M \setminus \operatorname{supp} b) \cup V$, and so we use sheaf property of C^k_N to glue $g \cdot b \in C^k(V)$ with $0 \in C^k(M \setminus \operatorname{supp} b)$ into a C^k-function $\tilde{g}$. By construction $\tilde{g}$ coincides with g on $\overline{W}_y$. By assumption $f^*(\tilde{g}) \in C^k_M(M)$. But

$$f^*(\tilde{g})|_{f^{-1}(W_y)} \equiv f^*(g)|_{f^{-1}(W_y)}.$$

Covering V by all possible W_y, we obtain the result. □

Example 3.15. Let $f : \mathbb{R} \to \mathbb{S}^1$ be the map $\alpha \mapsto (\cos\alpha, \sin\alpha)$. Then this map is smooth since for each of the four standard charts $\varphi^{\pm}_{0,1} : U^{\pm}_{0,1} \to]-1, 1[$ the composition of φ's with f is smooth. For example, for $\varphi^+_0 : \{x_0 > 0\} \to]-1, 1[$ the composition $\varphi^+_0 \circ f(\alpha) = \sin\alpha$ is smooth.

Example 3.16. (Hopf fibration). The natural isomorphism $\mathbb{C} \cong \mathbb{R}^2$ is a smooth map. We use it to identify

$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\},\ \mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1\}.$$

We now define $f : \mathbb{S}^3 \to \mathbb{S}^2$ by setting $f(z, w) = (2z\overline{w}, |z|^2 - |w|^2)$. This works since the second component is real and

$$|2z\overline{w}|^2 + (|z|^2 - |w|^2)^2 = (|z|^2 + |w|^2)^2 = 1.$$

If needed, one can write $z = x_0 + ix_1$, $w = x_0 + ix_2$ and get explicit formulas for the map p. We leave as an exercise to reflect on why it is smooth (can actually obtain it as a restriction of a smooth map $\mathbb{R}^4 \to \mathbb{R}^3$).

Note that if $p(z, w) = p(z', w')$ is only possible if $(z, w) = (\lambda z, \lambda w)$ where $\lambda = \exp(i\theta)$ (again, exercise). For this reason $p^{-1}(y) \cong \mathbb{S}^1$.

Notation 3.17. For M, N C^k-manifolds, denote $C^k(M, N)$ to be the set of C^k-maps. We can use 3.14.4 to quickly verify that the composition of C^k-maps is C^k, hence

$$C^k(M, N) \times C^k(N, L) \longrightarrow C^k(M, L), \quad (f, g) \mapsto g \circ f$$

is well defined: for $h \in C^k(L)$, $(g \circ f)^*(h) = f^*(g^*(h)) \in C^k(M)$.

Lemma 3.18. *Let M, N be C^k-manifolds. For $U \in \mathsf{Op}(M)$, consider $C^k(U, N) \subset \mathcal{F}(U, N)$. This defines a sheaf.*

Proof.

1. If $F : U \to N$ is C^k take $V \in \mathsf{Op}(U)$ and consider the restriction $F|_V : V \to N$. For each $f \in C^k(N)$ the composition $f \circ F$ belongs to $C^k_U(U)$. However, $f \circ (F|_V) = (f \circ F)|_V$ and since C^k_U is a presheaf, we have that $f \circ (F|_V)$ is C^k
2. This proves that $\{C^k(U, N)\}_{U \in \mathsf{Op}\, M}$ is a presheaf.
3. Let $U = \cup_i U_i$ and consider $F : U \to N$ such that $F|_{U_i} \in C^k(U_i, N)$. Take $f \in C^k(N)$. We have, by assumption, that

   $$(f \circ F)|_{U_i} = f \circ (F|_{U_i}) \in C^k(U_i) \text{ for each } U_i.$$

   Hence $f \circ F \in C^k(U)$ and this proves the sheaf property. □

3.3 The tangent space

Let M be a smooth manifold. Some questions that we will attempt to address:

1. Abstractly speaking, what is "tangent space"? Why should we care about it? Our answer will be: tangent space is related to linear approximation of smooth maps, and consists of abstract directional derivative operators of smooth functions.
2. How does one proceed to define differential of smooth maps?
3. When $M = \{f = 0\}$ (for "good enough" f), what is the relation between this abstract tangent space and the concrete one, $\ker df(x) = 0$?
4. What do all tangent spaces bundled together form? What is a tangent-vector valued function?

We put $k = \infty$ in this subsection, even though most statements are doable for $k \geq 2$. Our presentation is very close to Lee [4].

Let $\Omega \subset \mathbb{R}^n$. What does it mean to have a tangent vector at $p \in \Omega$? Our answer: directional derivatives:

$$f \in C^\infty(\Omega), v \in \mathbb{R}^n, \;\; \tilde{v}_p(f) := \left.\frac{d}{dt}\right|_{t=0} f(p + tv) = \sum_i v^i \frac{\partial f}{\partial x^i}(p)$$

(indices up! Physics notation).

Lemma 3.19. *The assignment $f \mapsto \tilde{v}_p(f)$ is an $\mathbb{R}$-linear map $C^\infty(\Omega) \to \mathbb{R}$ that satisfies:*

$$\tilde{v}_p(fg) = \tilde{v}_p(f)g(p) + f(p)\tilde{v}_p(g).$$

Proof. Leibniz rule for derivatives. □

Definition 3.20. A *derivation at* $p \in \Omega$ is an $\mathbb{R}$-linear map $X : C^\infty(\Omega) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$X(fg) = X(f)g(p) + f(p)X(g).$$

We denote $\mathrm{Der}_p(C^\infty(\Omega), \mathbb{R})$ the set of all derivations at p.

Lemma 3.21. *Let Ω be an open subset of $\mathbb{R}^n$, then*

1. *The set $\mathrm{Der}_p(C^\infty(\Omega), \mathbb{R})$ is an $\mathbb{R}$-vector space and the map $\mathbb{R}^n \to \mathrm{Der}_p(C^\infty(\Omega), \mathbb{R})$, $v \mapsto \tilde{v}_p$ is $\mathbb{R}$-linear.*
2. *If f is a constant function then $X(f) = 0$.*
3. *If $f(p) = g(p) = 0$, then $X(fg) = 0$.*

Proof.

1. The vector space structure is given by $(X+Y)(f) := X(f) + Y(f)$, $(\lambda X)(f) := \lambda(X(f))$. The linearity of the map $v \mapsto \tilde{v}_p$ can be verified using coordinate representation $\tilde{v}_p(f) = \sum_i v^i \frac{\partial f}{\partial x^i}(p)$:

$$(\widetilde{\lambda v + w})_p(f) = \sum_i (\lambda v^i + w^i)\frac{\partial f}{\partial x^i}(p) = \lambda \sum_i v^i \frac{\partial f}{\partial x^i}(p) + \sum_i w^i \frac{\partial f}{\partial x^i}(p).$$

2. Since $f = \lambda \cdot 1$ where 1 is the unity function on Ω it suffices to show $X(1) = 0$. Yet

$$X(1) = X(1 \cdot 1) = 1 \cdot X(1) + X(1) \cdot 1 = 2X(1)$$

and so $X(1)$=0.

3. $X(fg) = f(p)X(g) + X(f)g(p) = 0 + 0 = 0.$ □

Tangent space to $p \in \Omega$

Definition 3.22. The *tangent space* to Ω at p is defined as

$$T_p\Omega := \mathrm{Der}_p(C^\infty(\Omega), \mathbb{R}).$$

This is in fact a finite dimensional space.

Proposition 3.23. *Let Ω be a convex subset of $\mathbb{R}^n$. The map $\mathbb{R}^n \to \mathrm{Der}_p(C^\infty(\Omega), \mathbb{R})$, $v \mapsto \tilde{v}_p$ is an isomorphism. In particular, a basis of $T_p\Omega$ is given by*

$$\partial_1|_p := \left.\frac{\partial}{\partial x^1}\right|_p, ..., \partial_n|_p := \left.\frac{\partial}{\partial x^n}\right|_p,$$

where $\left.\frac{\partial}{\partial x^i}\right|_p(f) = \frac{\partial f}{\partial x^i}(p)$.

The notation will be further simplified: when the context is clear, we write $\partial_i \equiv \partial_i|_p$.

Proof. The statement relies on the first order Taylor formula for functions. Let $f \in C^2(\Omega)$, then

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x)$$

where $g_i \in C^1(\Omega)$.

This formula is proved as follows:

$$\begin{aligned} f(x) - f(p) &= \int_0^1 \frac{d}{dt}\left(f(tx + (1-t)p)\right) dt \\ &= \int_0^1 \sum_i \partial_i f(tx + (1-t)p) \cdot \frac{d}{dt}(tx^i + (1-t)p^i) dt \\ &= \sum_i (x^i - p^i) \int_0^1 \partial_i f(tx + (1-t)p) dt. \end{aligned}$$

We used convexity of Ω and Leibniz integral rule to assure the properties of $g_i(x) = \int_0^1 \partial_i f(tx + (1-t)p)dt$. Note that automatically $\partial_i f(p) = g_i(p)$.

Now, let $X \in T_p\Omega$. Then, denoting by x^i the function $x \mapsto x^i$,

$$X(f) = X(f(p) + \sum_i (x^i - p^i) g_i) = 0 + \sum_i \left(X(x^i - p^i) g_i(p) + (p^i - p^i) X(g_i) \right)$$

and so

$$X(f) = \sum_i X(x^i) g_i(p) = \sum_i X(x^i) \partial_i f(p).$$

Thus for any $X \in T_p\Omega$, there is always $v = (X(x^1), ..., X(x^n))$ in $\mathbb{R}^n$ such that $\tilde{v}_p = X$. This proves that $\mathbb{R}^n \to T_p\Omega$ is surjective.

If $\tilde{v}_p = 0$ then $v^i = \tilde{v}_p(x^i) = \sum_i v^i \partial_i (x^i)|_p = 0$. This proves that $\mathbb{R}^n \to T_p\Omega$ is injective. $\square$

Tangent space to $p \in M$

Let (M, C^∞_M) be a smooth manifold.

Definition 3.24. A *derivation at* $p \in M$ is an $\mathbb{R}$-linear map $X : C^\infty(M) \to \mathbb{R}$ that satisfies the abstract Leibniz rule:

$$X(fg) = X(f)g(p) + f(p)X(g).$$

We denote $T_p(M) \equiv \mathsf{Der}_p(C^\infty(M), \mathbb{R})$ the set of all derivations at p, and call it *the tangent space to M at p*.

Possible objections:

- it is "too much": why does it depend on $C^\infty(M)$ and not on a more local $C^\infty(U)$ for $p \in U \in \mathsf{Op}(M)$?
- it is "too little": global functions restrict to local functions, but not all local functions prolong to global functions.

By addressing the locality phenomenon we shall arrive at another, equivalent definition.

Lemma 3.25. *Let M be a smooth manifold, then*

1. *The set $T_pM = \mathrm{Der}_p(C^\infty(M), \mathbb{R})$ is an $\mathbb{R}$-vector space.*
2. *If f is a constant function on M then $X(f) = 0$.*
3. *If $f(p) = g(p) = 0$, then $X(fg) = 0$.*

Proof. Same as Lemma 3.21. $\square$

We will study T_pM by relating it to tangent spaces of other manifolds. Tangent vectors can be pushed forward using the pullback of functions:

3.3.1 The generalisation of differential

Definition 3.26. Let $F : (M, C^\infty_M) \to (N, C^\infty_N)$ be a smooth map and $p \in M$. The *pushforward along F at p*, or the *differential of F at p*, is the map $F_* : T_pM \to T_{F(p)}N$ defined as

$$X \in T_pM \ \mapsto F_*X \in T_{F(p)}N, \ F_*X(g) := X(F^*(g)) \text{ for } g \in C^\infty(N).$$

We shall sometimes interchange the notation F_*X, $F_*(X)$.

Example 3.27. If we have a smooth map $\mathbb{R}^n \supset \Omega \overset{F}{\to} \Theta \subset \mathbb{R}^m$ between two convexes, let us examine what it does to $\tilde{v}_p = \sum v^i \partial_i \in T_p\Omega$. Denoting $(x^1, ..., x^n)$ the points of $\mathbb{R}^n$ and $(y^1, ..., y^m)$ the points of $\mathbb{R}^m$, we have

$$F_*\tilde{v}_p(g) = \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_{x=p} g(F(x)) = \sum_{i,j} v^i \left(\frac{\partial}{\partial y^j}\bigg|_{y=F(p)} g(y)\right) \cdot \frac{\partial F^j}{\partial x^i}(p)$$

$$= \sum_{i,j} (J^j_i v^i)\partial_j g(F(p)),$$

where J^j_i is the Jacobi matrix for the map F at p. In other words, $F_*\tilde{v}_p = \widetilde{dF(p)(v)}$ where $dF(p) : \mathbb{R}^n \to \mathbb{R}^m$ is the ordinary differential map. We can conclude by saying that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "dF(p)"] \arrow[d, "\sim"'] & \mathbb{R}^m \arrow[d, "\sim"] \\
T_p\Omega \arrow[r, "F_*"'] & T_{F(p)}\Theta
\end{tikzcd}$$

For this reason, we shall often go back and forth between F_* and $dF(p)$ in the context of opens in $\mathbb{R}^n$.

Properties of pushforwards

Proposition 3.28. *Let $F : M \to N$ and $G : N \to K$ be two smooth maps, $p \in M$.*

1. *The map $F_* : T_pM \to T_{F(p)}N$ is well-defined and is $\mathbb{R}$-linear.*
2. *$(G \circ F)_* = G_* \circ F_*$ and $(\mathsf{id}_M)_* = \mathsf{id}_{T_pM}$ (this property is called functoriality).*
3. *If F is a diffeomorphism then F_* is an isomorphism.*

Proof.

1. If $X \in T_pM$ then

$$\begin{aligned} F_*X(fg) &= X(F^*(fg)) = X(F^*(f) \cdot F^*(g)) \\ &= F^*(f)(p) \cdot X(F^*(g)) + X(F^*(f)) \cdot F^*(g)(p) \\ &= f(F(x))F_*X(g) + F_*X(f)g(F(p)) \end{aligned}$$

so F_* is indeed well-defined. To check the linearity,

$$\begin{aligned} F_*(\lambda X + \mu Y)(f) &= (\lambda X + \mu Y)(f \circ F) = \lambda X(f \circ F) + \mu Y(f \circ F) \\ &= \lambda F_*X(f) + \mu F_*Y(f). \end{aligned}$$

2. $(G \circ F)_*X(f) = X(f \circ G \circ F) = F_*X(f \circ G) = G_*(F_*X(f))$. $X(f \circ \mathsf{id}_M) = X(f)$.
3. A diffeomorphism is a map $F : M \to N$ that admits smooth inverse $G : N \to M$. Since $F \mapsto F_*$ preserves compositions and identities, it sends mutually inverse maps to mutually inverse maps. ☐

The following proposition formalises the fact that T_pM is local:

Proposition 3.29. *Let M be a smooth manifold and $U \in \mathsf{Op}(M)$. Denote $i : U \hookrightarrow M$ the natural inclusion map. Then for each point $p \in U$, the map $i_* : T_pU \to T_pM$ is an isomorphism.*

Proof.

1. First, prove: $f, g \in C^\infty(M)$ and $X \in T_pM$, if $h = f - g \equiv 0$ on some neighbourhood V of p, we have $X(f) = X(g)$.

   Corollary 3.5 implies that there exists $b \in C^\infty(M)$ which is equal to 1 on $\overline{W} \subset V$ and has support contained in V. Denote $u = 1 - b$. By construction $u \equiv 1$ on $\mathsf{supp}\, h$ so $u \cdot h = h$ everywhere. But $u(p) = h(p) = 0$ and so

$$X(h) = X(uh) = u(p)X(h) + X(u)h(p) = 0.$$

2. The injectivity of i_*. Let $X \in T_pU$ and $i_*X(f) = 0$ for all $f \in C^\infty(M)$. However if $g \in C^\infty(U)$, using a bump function we can construct $\tilde{g} \in C^\infty(M)$ that agrees with g on some $p \in W \subset U$. Because of 1. we have:

$$X(g) = X(\tilde{g}|_U) = X(\tilde{g} \circ i) = i_*X(\tilde{g}) = 0$$

   since this is true for all g, we conclude that $X = 0$.

3. The surjectivity of i_*. Let $Y \in T_pM$. Define $\tilde{Y} \in T_pU$ by setting $\tilde{Y}(f) = Y(\tilde{f})$ for an extension $\tilde{f} : M \to \mathbb{R}$ that agrees with $f : U \to \mathbb{R}$ on some neighbourhood of p. Because of 1. this definition does not depend on choice of $\tilde{f}$.

   If we have $f, g \in C^\infty(U)$, we can choose $\widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g}$ as an extension of the product. Thus

$$\tilde{Y}(f \cdot g) = Y(\widetilde{f \cdot g}) = \tilde{f}(p)Y(\tilde{g}) + Y(\tilde{f})\tilde{g}(p) = f(p)\tilde{Y}(g) + \tilde{Y}(f)g(p).$$

   It only remains to see if $i_*\tilde{Y} = Y$. For this,

$$i_*\tilde{Y}(f) = \tilde{Y}(f|_U) = Y(\widetilde{f|_U}) = Y(f)$$

   the last equality uses 1. since the extension $\widetilde{f|_U}$ is equal to f on some neighbourhood of p. □

As a consequence, we shall identify $T_pU \cong T_pM$ along the canonical inclusion map $i : U \hookrightarrow M$.

Corollary 3.30. *Let $\Omega \subset \mathbb{R}^n$ be any open subset and $p \in \Omega$. Then the map $\mathbb{R}^n \to T_p\Omega$, $v \mapsto \tilde{v}_p$, is an isomorphism. Moreover, for any smooth map $\Omega \xrightarrow{F} \Theta$, with $\Theta \in \mathsf{Op}(\mathbb{R}^m)$, the following diagram commutes*

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "dF(p)"] \arrow[d, "\sim"'] & \mathbb{R}^m \arrow[d, "\sim"] \\ T_p\Omega \arrow[r, "F_*"'] & T_{F(p)}\Theta \end{tikzcd}$$

with the upper map denoting the ordinary differential of F.

Proof. The pairs $(\Omega, C^\infty_\Omega)$ and $(\Theta, C^\infty_\Theta)$ are smooth manifolds that admit covers by open balls as charts. Let $B(p, \delta) \subset \Omega$ be such that $F(B(p, \delta)) \subset B(F(p), \varepsilon)$. The following diagram commutes:

$$\begin{tikzcd} T_pB(p,\delta) \arrow[r, "F_*"] \arrow[d, "\sim"'] & T_{F(p)}B(F(p),\varepsilon) \arrow[d, "\sim"] \\ T_p\Omega \arrow[r, "F_*"'] & T_{F(p)}\Theta. \end{tikzcd}$$

Since balls are convex, the rest follows from Proposition 3.23 and Example 3.27.

Example: $\mathbb{S}^n$

Example 3.31. 1. Recall that $\mathbb{S}^n = F^{-1}(1)$ for $F(x) = \|x\|^2$. We can view F as C^∞ function from $M = \mathbb{R}^{n+1} \setminus 0$ to $\mathbb{R}_{>0}$.

2. Denote $i : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus 0 = M$ the canonical inclusion (smooth). Let $p \in \mathbb{S}^n$. We want to relate $T_p\mathbb{S}^n$ to a subspace in $T_{i(p)}M \cong \mathbb{R}^{n+1}$.

3. Note that $F \circ i$, $x \mapsto F(x) = 1$ is a constant function. For this reason $F_* i_*(X)(f) = X(f(1)) = 0$ for all $X \in T_p\mathbb{S}^n$ and $f \in C^\infty(M)$. This means that $\mathsf{im}\, i_* \subset \mathsf{ker}\, F_*$.

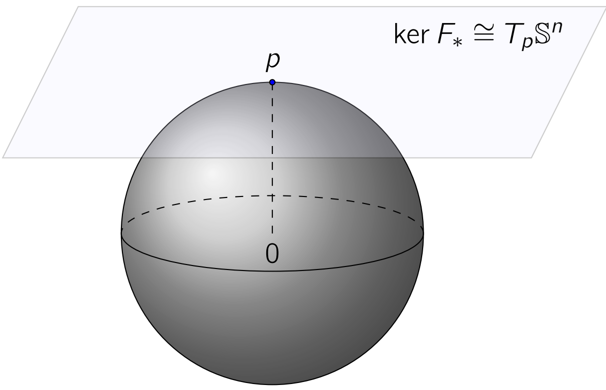
4. Recall $u : M \to \mathbb{S}^n$, $x \mapsto x/\|x\|$ (smooth). Since $u \circ i = \mathsf{id}_{\mathbb{S}^n}$, the composition $u_* \circ i_*$ is also an identity on $T_p\mathbb{S}^n$. This proves that i_* is injective (immersion at p), meaning $T_p\mathbb{S}^n \cong \mathsf{im}\, i_*$.

5. We now claim that $\mathsf{im}\, i_* = \mathsf{ker}\, F_*$.

6. First, $\mathsf{dim}\,\mathsf{im}\, i_* = \mathsf{dim}\, T_p\mathbb{S}^n = n$. Second, by Corollary 3.30, $\mathsf{dim}\,\mathsf{ker}\, F_* = \mathsf{dim}\,\mathsf{ker}\, dF(p)$.

7. However, $dF(p) = 2(p^0, ..., p^n)$ is a nonzero linear form, and so $\mathsf{dim}\,\mathsf{ker}\, dF(p) = (n+1) - 1 = n$.

8. We conclude that $T_p\mathbb{S}^n \cong \mathsf{ker}\, dF(p)$. This is natural since
$$\mathsf{ker}\, dF(p) = \{v \in \mathbb{R}^{n+1} \,|v^0 p^0 + ... + v^n p^n = 0\}$$
is the hyperplane orthogonal to the vector $p \in \mathbb{S}^n$.



3.3.2 Coordinates

Local coordinate presentation

Corollary 3.32. *Let M be a smooth manifold of dimension n. Then for each $p \in M$, the vector space T_pM is finite dimensional, of dimension n.*

Proof. Choose a smooth chart (U, φ) containing p. Then the following chain of isomorphisms suffices:
$$\mathbb{R}^n \cong T_{\varphi(p)}\varphi(U) \xleftarrow[\varphi_*]{\sim} T_pU \cong T_pM.$$
Note that the identification above depends, in particular, on the choice of φ. ☐

Notation 3.33. In the situation above, we can use $\varphi : U \cong \Omega$ to write a particular basis for T_pM. If $q \in U$ and $\varphi(q) = (x^1(q), ..., x^n(q))$, then we denote
$$\partial_i|_p \equiv \left.\frac{\partial}{\partial x^i}\right|_p := \varphi_*^{-1} \left.\frac{\partial}{\partial x^i}\right|_{x=\varphi(p)}.$$

This set is a basis of T_pU that we canonically identify with T_pM.

I personnally find that dropping φ from the notation is a bit confusing and misleading. The partial derivatives do not exist in a given way, the space T_pM usually has no canonical basis!

I rather prefer the following. Let $V \in T_pM$, then to say that V has a decomposition with respect to basis given by φ, we write

$$\varphi_* V = \sum V^i \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}.$$

The bar notation $|_{\varphi(p)}$ signifying the point will sometimes be implicitly understood; in fact, it is entirely possible to think of $\partial/\partial x^i$ as of a *vector field*, something that we will discover in future lectures.

Question: how does that depend on φ? In other words, if we have two chart structures φ, ψ and a decomposition of $\varphi_* V$, how to compute $\psi_* V$?

Coordinate transformations

Assume that U has two chart maps, $\varphi : U \xrightarrow{\sim} \Omega$ and $\psi : U \xrightarrow{\sim} \Theta$ (this can happen for example if U is an intersection of two charts). The smoothness of M guarantees that $J := \psi \circ \varphi^{-1} : \Omega \to \Theta$ is a smooth map.

Let $p \in U$ and

- for $q \in U$ denote $\varphi(q) = (x^1(q), ..., x^n(q))$ and $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$ the basis of $T_{\varphi(p)}\Omega$,
- for $q \in U$ denote $\psi(q) = (y^1(q), ..., y^n(q))$ and $\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$ the basis of $T_{\psi(p)}\Theta$.

The following two diagrams are commutative:

$$\begin{tikzcd} & U & \\ \Omega \arrow[ur, "\varphi^{-1}"] \arrow[rr, "J = \psi \circ \varphi^{-1}"'] & & \Theta, \arrow[ul, "\psi^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} & T_pM & \\ T_{\varphi(p)}\Omega \arrow[ur, "\varphi_*^{-1}"] \arrow[rr, "J_*"'] & & T_{\psi(p)}\Theta, \arrow[ul, "\psi_*^{-1}"'] \end{tikzcd}$$

and thus we will know everything once we understand the action of J_*. The latter is expressed via the differential of J using Corollary 3.30.

The usual notation is as follows. The map $J : \Omega \to \Theta$ allows to write $y^i(x) := y^i(\psi(\varphi^{-1}(x)))$. The Jacobi matrix is the given by

$$J^j_i = \frac{\partial y^j}{\partial x^i}(\varphi(p))$$

because of Corollary 3.30,

$$J_* \frac{\partial}{\partial x^i} = J_* \tilde{e}_i = \widetilde{\sum_j J^j_i e_j} = \sum_j \frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j},$$

where e_i are the standard basis vectors of $\mathbb{R}^n$. Thus the transformation rule for the basis vectors of T_pM is simply the chain rule. Similarly,

$$\varphi_*^{-1} \frac{\partial}{\partial x^i} = \psi_*^{-1} J_* \frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i}(\varphi(p)) \psi_*^{-1} \frac{\partial}{\partial y^j}.$$

If we were to drop φ and ψ as in Notation 3.33 we could write:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_j \frac{\partial y^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial y^j}\right|_p.$$

Again, I find it a little misleading due to the dependence on chart maps.

For $V \in T_pM$, assume that $\varphi_* V = \sum_i X^i \frac{\partial}{\partial x^i}\big|_{\varphi(p)}$. Then

$$\psi_* V = \psi_*(\varphi_*^{-1}\varphi_* V) = J_*\varphi_* V.$$

so if $\psi_* V = \sum_i Y^i \frac{\partial}{\partial y^i}\big|_{\psi(p)}$, we have

$$\sum_i Y^i \frac{\partial}{\partial y^i} = J_* \sum_i X^i \frac{\partial}{\partial x^i} = \sum_{i,j} X^i \frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}$$

and thus we derive the usual transformation rule for the coefficients:

$$Y^i = \sum_j X^j \frac{\partial y^i}{\partial x^j}(\varphi(p)).$$

The mnemoinc rule here is that x's are summed with X's and the "lowered index" of the derivative is always summed with the "upper index" of the tangent vector coefficient.

Example 3.34.

1. Consider $i : \mathbb{S}^1 \to \mathbb{R}^2 = \{(x, y)\}$. We have shown above that for each point $p \in \mathbb{S}^1$ the map $i_*(p) : T_p\mathbb{S}^1 \to T_p\mathbb{R}^2$ injectively maps $T_p\mathbb{S}^1$ onto the subspace of $T_p\mathbb{R}^2$ consisting of all derivations $a\partial_x|_p + b\partial_y|_p$ such that $ax_p + by_p = 0$ where $p = (x_p, y_p)$.
2. As such we can write $T_p\mathbb{S}^1 \cong \mathsf{Span}(x_p\partial_y - y_p\partial_x)$. Let $V \in T_p\mathbb{S}^1$ be the unique vector such that $i_*V = x_p\partial_y - y_p\partial_x$. How to represent it in a coordinate chart?
3. For instance, assume that $x_p > 0$. Let $\varphi \equiv \varphi_x^+ : \mathbb{S}^1 \cap \{x > 0\} \to]-1, 1[$ be the coordinate projection $(x, y) \mapsto y$. I want to compute $\varphi_* V$. For this, I use the normalisation map $u : \mathbb{R}^2 \setminus 0 \to \mathbb{S}^1$ and $u \circ i = \mathsf{id}_{\mathbb{S}^1}$:

$$\varphi_* V = (\varphi \circ u \circ i)_* V = (\varphi \circ u)_* \circ i_* V.$$

4. The map $\varphi \circ u$ was encountered before, we called it Φ. It acts as

$$\Phi(x, y) = \frac{y}{\sqrt{x^2 + y^2}}.$$

   We can then compute its differential and use it to conclude.

5. Let us compute the differential

$$d\Phi(x, y) = \frac{dy}{\sqrt{x^2 + y^2}} - \frac{x dx + y dy}{(\sqrt{x^2 + y^2})^3},$$

   where I denote dx, dy the canonical dual basis: $dx(a, b) = a$, $dy(a, b) = b$.

6. Note that $xdx + ydy$ is always zero on $\operatorname{im} i_*$ since this is the differential of the equation $x^2 + y^2 - 1$. Thus at $p \in \mathbb{S}^1 \cap \{x > 0\}$, we have

$$d\Phi(x_p, y_p)(-y_p, x_p) = x_p.$$

In terms of derivations, we can write that V, $i_*V = x_p\partial_y|_p - y_p\partial_x|_p$ becomes

$$\varphi_* V = x_p \partial_t|_{t=y_p}$$

where $t \in]-1, 1[$ is the coordinate in the image of φ.

Coordinate expression for pushforwards

Let $F : M \to N$ be smooth. Let $p \in M$. Take an open chart (V, ψ) containing $F(p) \in N$ and find an open chart (U, φ) containing $p \in M$ such that $F(U) \subset V$ (take intersections if needed).

Denote $\tilde{F} := \psi \circ F \circ \varphi^{-1}$. The following diagrams commute:

$$\begin{tikzcd} \Omega \arrow[r, "\tilde{F}"] & \Theta \\ U \arrow[u, "\varphi", "\sim"'] \arrow[r, "F|_U"] & V \arrow[u, "\psi"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} T_{\varphi(p)}\Omega \arrow[r, "\tilde{F}_*"] & T_{\psi(F(p))}\Theta \\ T_pM \arrow[u, "\varphi_*", "\sim"'] \arrow[r, "F_*"] & T_{F(p)}N \arrow[u, "\psi_*", "\sim"'] \end{tikzcd}$$

If we denote $(x^1, ..., x^m)$ the coordinates in Ω and $(y^1, ..., y^n)$ the coordinates in Θ, then repeating the same calculus as in Example 3.27, we get

$$\tilde{F}_* \frac{\partial}{\partial x^i} = \sum_j \frac{\partial \tilde{F}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}.$$

Here $\left(\frac{\partial \tilde{F}^j}{\partial x^i}(\varphi(p))\right)_{i=1,m}^{j=1,n}$ is the Jacobi matrix of the map $\tilde{F} : \Omega \to \Theta$ at $\varphi(p)$.

If $V \in T_pM$ is such that $\varphi_* V = \sum X^i \frac{\partial}{\partial x^i}|_{\varphi(p)}$, then, just like before,

$$\psi_*(F_*V) = \tilde{F}_*(\varphi_*V)$$

And so if $\psi_*(F_*V) = \sum Y^i \frac{\partial}{\partial x^i}|_{\psi(F(p))}$ we can repeat the same analysis as before to conclude that

$$\sum_i Y^i \frac{\partial}{\partial y^i}\Big|_{\psi(F(p))} = \tilde{F}_* \left(\sum_j X^j \frac{\partial}{\partial x^j}\Big|_{\varphi(p)} \right) = \sum_{i,j} X^j \frac{\partial \tilde{F}^i}{\partial x^j}(\varphi(p)) \frac{\partial}{\partial y^i}\Big|_{\psi(F(p))}$$

and so

$$Y^i = \sum_j \frac{\partial \tilde{F}^i}{\partial x^j}(\varphi(p)) X^j.$$

The mnemonic rules remain the same.

3.3.3 Miscellaneous

Alternative definition: germs

The canonical identification suggests that what matters for T_pM are only functions on some small neighbourhood of $p \in M$. Indeed, if we have

$$M \supset U_0 \supset U_1 \supset ... \supset U_n \supset ...$$

where all U_i contain x then

$$T_pM \cong T_pU_0 \cong T_pU_1 \cong ... \cong T_pU_n \cong ...$$

To formalise that define

Definition 3.35. The $\mathbb{R}$-*algebra of smooth germs at* p, C_p^∞, is defined as follows:

1. An element of C_p^∞ is an equivalence class $[f, U] \equiv [(f, U)]$ where $p \in U \in \mathsf{Op}\, X$ and $f \in C^\infty(U)$. We put $(f, U) \sim (g, V)$ if $f \equiv g$ on $U \cap V$.

2. The algebra structure is given by

$$[f, U] + [g, V] = [f + g, U \cap V], \quad \lambda[f, U] = [\lambda f, U],$$
$$[f, U] \cdot [g, V] = [f \cdot g, U \cap V].$$

Just as before, denote $\mathsf{Der}(C_p^\infty, \mathbb{R})$ the set of $\mathbb{R}$-linear maps $X : C_p^\infty \to \mathbb{R}$ that satisfy the rule

$$X([f, U] \cdot [g, V]) = f(p)X([g, V]) + X([f, U])g(p),$$

and this definition makes sense since all representatives of a germ take the same value at p. The set $\mathsf{Der}(C_p^\infty, \mathbb{R})$ is a vector space, same proof.

Proposition 3.36. *Let M be a smooth manifold and $p \in M$. The map*

$$\mathsf{Der}(C_p^\infty, \mathbb{R}) \to T_pM, \ X \mapsto \tilde{X},$$

that takes X to a derivation defined by $\tilde{X}(f) := X([f, M])$, is an isomorphism. Consequently, one can define T_pM as derivations of germs.

Proof. Usual bump function games to construct the inverse. Elaborate in TD. $\square$

Tangent vectors to curves

Let M be a manifold, $a \neq b$ and $\gamma :]a, b[\to M$ a smooth map (makes sense, since $]a, b[$ is a smooth 1-dimensional manifold). Let $t_0 \in]a, b[$ and $p = f(t_0)$.

Denoting by $\frac{d}{dt}\big|_{t_0}$ the tangent basis vector to $]a, b[$ at t_0, we define

$$\gamma'(t_0) := \gamma_* \frac{d}{dt}\bigg|_{t_0} \in T_pM.$$

If (U, φ) is some coordinate chart that contains the image of some neighbourhood of t_0, then we can write $\varphi \circ \gamma(t) = (\gamma^1(t), ..., \gamma^n(t))$ and compute:

$$\varphi_* \gamma'(t_0) = \sum_i \frac{d\gamma^i}{dt}(t_0) \cdot \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}.$$

Lemma 3.37. *Every $X \in T_pM$ is a tangent vector to some curve.*

Proof. Find a coordinate chart (U, φ) centred at p. Let $X = \sum_i X^i \partial_i$. Define $\gamma :]-\varepsilon, \varepsilon[\to M$ by setting $\gamma(t) := \varphi^{-1}((tX^1, ..., tX^n))$. If ε is small enough γ is well defined, $\gamma(0) = p$ and $\varphi_* \gamma'(t_0) = \sum X^i \partial_i$. $\square$

4 Submersions, immersions, submanifolds

We can use our generalisation of the differential to study maps of smooth manifolds.

Definition 4.1. Let $F : M \to N$ be a smooth map.

1. F is *of rank* k at $p \in M$ if for $F_* : T_pM \to T_{F(p)}N$, $\dim \operatorname{im} F_* = k$.
2. In particular, F is an *immersion at* p if $k = \dim M$ (in other words, F_* is injective), and
3. F is a *submersion at* p if $k = \dim N$ (in other words, F_* is surjective).

We shall say that

1. F is *of constant rank* k,
2. an *immersion*,
3. or a *submersion*,

if the mentioned quality is true at each $p \in M$.

Remark 4.2. True stories:

1. F immersion (at p) $\implies$ $\dim M \leq \dim N$,
2. F submersion (at p) $\implies$ $\dim M \geq \dim N$,
3. F of rank k (at p) $\implies$ $\dim N \leq k \leq \dim M$.

Some examples

Example 4.3. 1. Any inclusion of an open $U \hookrightarrow M$ is of (maximal) constant rank $k = \dim M$.

2. Let $\pi : M \to N$ be a smooth map such that there exists $U \subset N$ and $s \in C^\infty(U, M)$ satisfying $\pi \circ s = \mathrm{id}_U$. Then for each $p \in U$, we have
$$s_* : T_pN \to T_{s(p)}M, \quad \pi_* : T_{s(p)}M \to T_pN, \quad \pi_* \circ s_* = \mathrm{id}_{T_pN}\,.$$
This implies that s is an injective immersion on U and π a surjective submersion on $s(U)$.

3. In particular, Example 3.31 provided us with two smooth maps $u : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$ and $i : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus 0$ that satisfy $u \circ i = \mathrm{id}_{\mathbb{S}^n}$. We conclude that i is an injective immersion and u is a surjective submersion.

4. The map $F : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, t^3)$ is injective (its image is the cubic graph) and is an immersion. Indeed, $dF(t) = {}^t(1, 3t^2)$ is never zero. The map F is also a homeomorphism on its image.

5. The map $G : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^3$ is surjective (in fact, bijective), but is not a submersion (or immersion), for $dG(0) = 0$ as a linear map.

6. The map $f : \mathbb{R} \to \mathbb{S}^1$, $\alpha \mapsto (\cos\alpha, \sin\alpha)$ of Example 3.15 is a surjective submersion.

7. Let $n \geq k$. The projection map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ that sends $(x_1, ..., x_n) \to (x_1, .., x_k)$ is a submersion: the differential $d\pi(x)$ is of maximal rank k at each $x \in \mathbb{R}^n$. π is also surjective.

8. Let $n \leq k$. The inclusion map $\iota : \mathbb{R}^n \to \mathbb{R}^k$ that sends $(x_1, ..., x_n) \to (x_1, .., x_n, 0, ..., 0)$ is an immersion: the differential $d\iota(x)$ is of maximal rank n at each $x \in \mathbb{R}^n$. ι is also a homeomorphism on its image.

9. The map $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is a surjective submersion. The map $\mathbb{S}^n \to \mathbb{RP}^n$ is also surjective, of constant rank n.

Let us also present some examples without going into detail.

10. The map $\mathbb{R}^2 \to \mathbb{T}$ is surjective of constant rank 2.

11. The map $\mathsf{GL}_2^+(\mathbb{R}) \to \mathsf{SL}_2(\mathbb{R})$ defined by sending M to $\frac{1}{\sqrt{\det M}} M$ is a surjective submersion.

12. Define $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^3$ by the following rule:

$$\tilde{F}(x, y) = ((2 + \cos 2\pi x) \cos 2\pi y, (2 + \cos 2\pi x) \sin 2\pi y, \sin 2\pi x).$$

One can check that $\tilde{F}$ is an immersion. It is not injective, but it respects the torus equivalence relation on $\mathbb{R}^2$: $\tilde{F}(x + n, y + m) = \tilde{F}(x, y)$ This can be used to induce a smooth map $F : \mathbb{T} \to \mathbb{R}^3$ that is an immersion and a homeomorphism onto its image.

4.1 Local form of sub/immersions

Local diffeomorphisms

Let $F : M \to N$ be C^∞ and $p \in M$. Consider the case when $\dim M = \dim N$.

Theorem 4.4. *In the situation above, the following are equivalent:*

1. $F_* : T_pM \to T_{F(p)}N$ *is a bijection at* p.
2. F *is a* local diffeomorphism *at* p*: there exists* $U \in \mathsf{Op}\, M$ *containing* p *such that* $F(U)$ *is open and* $F|_U : U \to F(U)$ *is a diffeomorphism.*

We can illustrate the second property via the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "F|_U"', "\sim"] & M \arrow[d, "F"] \\ F(U) \arrow[r, hook] & N. \end{tikzcd}$$

Proof. If 2. is valid, then the diagram above gives, using Proposition 3.29:

$$\begin{tikzcd} T_pU \arrow[r, hook, "\sim"] \arrow[d, "(F|_U)_*"', "\sim"] & T_pM \arrow[d, "F_*"] \\ T_{F(p)}F(U) \arrow[r, hook, "\sim"'] & T_{F(p)}N, \end{tikzcd}$$

so the linear map $F_* : T_pM \to T_{F(p)}N$ can only be a bijection.

Let us now understand why 1. implies 2.. We can take a chart V containing $F(p)$. The set $F^{-1}(V)$ then contains p, and let U be some open chart containing p. Then $W = F^{-1}(V) \cap U$ is also a chart and we have the following commutative diagram:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "F|_W"'] & M \arrow[d, "F"] \\ V \arrow[r, hook] & N. \end{tikzcd}$$

Let $\varphi : W \xrightarrow{\sim} \Omega$ and $\psi : V \xrightarrow{\sim} \Theta$ be the corresponding chart maps. If we denote $\bar{F} = \psi \circ F \circ \varphi^{-1}$, the following diagram commutes:

$$\begin{tikzcd} \Omega \arrow[d, "\bar{F}"'] & W \arrow[l, "\varphi"', "\sim"] \arrow[r, hook] \arrow[d, "F|_W"] & M \arrow[d, "F"] \\ \Theta & V \arrow[l, "\sim"', "\psi"] \arrow[r, hook] & N. \end{tikzcd}$$

Moreover, the differential $d\bar{F}(\varphi(p))$ of the smooth map $\bar{F} : \Omega \to \Theta$ is invertible. Certainly such maps must have been studied in analysis?

Theorem 4.5. ***(Inverse function theorem [3])*** *Let Ω, Θ be two opens in $\mathbb{R}^n$, $f : \Omega \to \Theta$ a C^1-map, and let $df(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a bijection for some $x \in \Omega$. Then f is a local diffeomorphism: there is an open subset $U \subset \Omega$ that contains x, C^1-map $g : f(U) \to U$ such that g is inverse to $f|_U : U \to f(U) \subset \Theta$.* $\square$

This theorem may have been encountered already. If not, no need to worry: in fact, its proof is not needed to be remembered in order to use it!

We apply Theorem 4.5 to $\bar{F}$. Thus there exists $U \subset \Omega$ such that $\bar{F}|_U$ is a diffeomorphism and the diagram below commutes:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\bar{F}|_U"'] & \Omega \arrow[r, "\varphi^{-1}", "\sim"'] \arrow[d, "\bar{F}"'] & W \arrow[r, hook] \arrow[d, "F|_W"] & M \arrow[d, "F"] \\ \bar{F}(U) \arrow[r, hook] & \Theta \arrow[r, "\sim", "\psi^{-1}"'] & V \arrow[r, hook] & N. \end{tikzcd}$$

We collapse this diagram into a square by taking $\mathcal{U} = \varphi^{-1}(U)$ and using all the inclusions:

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d, "F|_{\mathcal{U}}"', "\sim"] & M \arrow[d, "F"] \\ F(\mathcal{U}) \arrow[r, hook] & N. \end{tikzcd}$$

This completes the proof. $\square$

We can conclude the following generalisation of a classical result from analysis:

Corollary 4.6. *Let $F : M \to N$ be a smooth map. If F is a bijection and for each $p \in M$, the map $F_* : T_pM \to T_{F(p)}N$ is also a bijection, then F is a diffeomorphism.*

Proof. We have the inverse $F^{-1} : N \to M$. Since our previous theorem implies that F is a local diffeomorphism, for each $y \in N$ there exists an open neighborhood V_y such that $F^{-1}|_{V_y}$ is smooth (uniqueness of inverses). Since $N = \cup_y V_y$ and smooth maps to M form a sheaf (Lemma 3.18), we conclude that F^{-1} is smooth. $\square$

Canonical submersion and immersion theorems

We have seen that $\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad (x_1, ..., x_m) \to (x_1, ..., x_n) \quad (m \geq n)$ is a submersion, and $\iota : \mathbb{R}^m \to \mathbb{R}^n, \quad (x_1, ..., x_m) \to (x_1, ..., x_m, 0, ..., 0) \quad (m \leq n)$ is an immersion.

Theorem 4.7. ***(canonical submersion/immersion)*** *Let $F : M \to N$ be a smooth map, $m = \dim M$, $n = \dim N$.*

1. *If F is a submersion at p, then there exist charts $\varphi : U \xrightarrow{\sim} \Omega$ containing p and $\psi : V \xrightarrow{\sim} \Theta$ containing $F(p)$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "\varphi", "\sim"'] \arrow[d, "F"'] & \Omega \arrow[d, "\pi"] \\ V \arrow[r, "\sim", "\psi"'] & \Theta. \end{tikzcd}$$

   In other words, F looks locally like a projection.

2. *If F is an immersion at p, then there exist charts $\varphi : U \xrightarrow{\sim} \Omega$ containing p and $\psi : V \xrightarrow{\sim} \Theta$ containing $F(p)$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "\varphi", "\sim"'] \arrow[d, "F"'] & \Omega \arrow[d, "\iota"] \\ V \arrow[r, "\sim", "\psi"'] & \Theta. \end{tikzcd}$$

   In other words, F looks locally like a subspace inclusion.

Proof of the submersion part

1. Given any permutation $\sigma : \{1, ..., n\} \to \{1, ..., n\}$, note that the induced map

$$\sigma^* : \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, ..., x_n) \mapsto (x_{\sigma(1)}, ..., x_{\sigma(n)})$$

   is a linear map and hence a diffeomorphism. If we have any chart $\psi : V \xrightarrow{\sim} \Theta$, then we can compose ψ with $\sigma^*|_\Theta$ to obtain a new chart $\psi^\sigma : V \xrightarrow{\sim} \sigma^*(\Theta)$.

2. Now we take any charts $\varphi : U \xrightarrow{\sim} \Omega$ containing p and $\psi : V \xrightarrow{\sim} \Theta$ containing $F(p)$ (we have shown how to construct them in Theorem 4.4):

$$\begin{tikzcd} U \arrow[r, "\varphi", "\sim"'] \arrow[d, "F"'] & \Omega \arrow[d, "\bar{F}"] \\ V \arrow[r, "\sim", "\psi"'] & \Theta. \end{tikzcd}$$

The map $\bar{F} = \psi \circ F \circ \varphi^{-1}$, by assumption, has the property that its differential is of rank n. This means that the rectangular matrix

$$J^i_j = \frac{\partial \bar{F}^i}{\partial x^j}(\varphi(p)), \quad 1 \le i \le n, \quad 1 \le j \le m,$$

has rank n ($n \le m$). Here $(x^1, ..., x^m)$ denote coordinates in Ω.

3. By using 1. if necessary, we can permute the coordinates $(y^1, ..., y^n)$ of Θ diffeomorphically. Without loss of generality, we can thus assume that the *square* matrix

$$J^i_j = \frac{\partial \bar{F}^i}{\partial x^j}(\varphi(p)), \quad 1 \le i \le n, \quad 1 \le j \le n,$$

is invertible.

4. Construct the following map:

$$G : \Omega \to \mathbb{R}^m, \quad x = (x^1, ..., x^m) \mapsto (\bar{F}^1(x), ..., \bar{F}^n(x), x^{n+1}, ..., x^m).$$

If we denote by $\det dG(\varphi(p))$ the determinant of the Jacobian matrix of G at $\varphi(p)$, we note that $\det dG(\varphi(p)) = \det J \neq 0$. This allows us to use Theorem 4.5.

5. That is, there is $W \subset \Omega$ such that $G|_W : W \to G(W)$ admits a smooth inverse H.

6. Also note that

$$\bar{F}(x) = (\bar{F}^1(x), ..., \bar{F}^n(x)) = \pi \circ G(x).$$

This suggests that we should try for a chart the map $G \circ \varphi$!

7. Let us write this all down. Denote $U_1 := \varphi^{-1}(W)$ and $\Omega_1 := G(W)$. Define $\varphi_1 : U_1 \to \Omega_1$ as $\varphi_1 := G \circ \varphi|_{U_1}$. I claim that the following diagram commutes:

$$\begin{tikzcd} \Omega_1 \arrow[r, "\varphi_1^{-1}"] \arrow[d, "\pi"'] & U_1 \arrow[d, "F|_{U_1}"] \\ \Theta \arrow[r, "\psi^{-1}"'] & V. \end{tikzcd}$$

If this is true, we are done, as U_1 and V will provide the answer.

8. To prove this final claim, compute:

$$F \circ \varphi_1^{-1} = F \circ \varphi^{-1} \circ G^{-1} = \psi^{-1} \circ \bar{F} \circ H = \psi^{-1} \circ \pi \circ G \circ H = \psi^{-1} \circ \pi.$$

That ends the submersion part.

Proof of the immersion part

0. The proof is roughly the same but will not be presented in class.

1. Fast forward to the following situation:

$$\begin{tikzcd} U \arrow[r, "\varphi", "\sim"'] \arrow[d, "F"'] & \Omega \arrow[d, "\bar{F}"] \\ V \arrow[r, "\sim", "\psi"'] & \Theta. \end{tikzcd}$$

The map $\bar{F} = \psi \circ F \circ \varphi^{-1}$ now has the property that

$$J^i_j = \frac{\partial \bar{F}^i}{\partial x^j}(\varphi(p)), \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

has rank m ($m \leq n$). Here $(x^1, ..., x^m)$ denote coordinates in Ω.

2. Without loss of generality, we can thus assume that the *square* matrix

$$J^i_j = \frac{\partial \bar{F}^i}{\partial x^j}(\varphi(p)), \quad 1 \leq i \leq m, \quad 1 \leq j \leq m$$

has rank m.

3. Consider the map

$$G : \Omega \times \mathbb{R}^{n-m} \to \mathbb{R}^n,$$
$$(x^1, ..., x^m, y^1, ..., y^{n-m}) \mapsto (\bar{F}^1(x), ...\bar{F}^m(x), \bar{F}^{m+1}(x) + y^1, ..., \bar{F}^n(x) + y^{n-m}).$$

If we compute its Jacobian matrix it looks like as follows:

$$\begin{pmatrix} \partial_1 \bar{F}^1 & ... & \partial_m \bar{F}^1 & 0 & 0 & ... & 0 \\ ... & & ... & & & ... & \\ \partial_1 \bar{F}^m & ... & \partial_m \bar{F}^m & 0 & 0 & ... & 0 \\ \partial_1 \bar{F}^{m+1} & ... & \partial_m \bar{F}^{m+1} & 1 & 0 & ... & 0 \\ ... & & ... & & & ... & \\ \partial_1 \bar{F}^n & ... & \partial_m \bar{F}^n & 0 & 0 & ... & 1 \end{pmatrix}$$

Here $\partial_i = \frac{\partial}{\partial x^i}$ (only means x-derivatives). If we evaluate this Jacobian at $(\varphi(p), 0)$, using the block-diagonal property of the determinant we conclude that $dG((\varphi(p), 0))$ is invertible.

4. Note also that $G \circ \iota : \Omega \to \mathbb{R}^n$ is in fact equal to $\bar{F}$ (corresponds to setting y's to zero).

5. By Theorem 4.5 there is a subset $W \subset \Omega \times \mathbb{R}^{n-m}$ on which G is invertible, with inverse $H : G(W) \to W$. By shrinking W in the y-direction we can assume that $G(W) \subset \Theta$.

6. Let $\Omega_1 := \iota^{-1}(W) \subset \Omega$ and $U_1 := \varphi^{-1}(\Omega_1)$. Denote also $\varphi_1 := \varphi|_{U_1}$. Then $\varphi : U_1 \xrightarrow{\sim} \Omega_1$ is a chart containing p.

7. Also let $\Theta_1 := W$, $V_1 := \psi^{-1}(G(W))$ and $\psi_1 : V_1 \to \Theta_1$ by $\psi_1 = H \circ \psi : V_1 \to G(W) \to W = \Theta_1$. I claim that the following diagram commutes:

$$\begin{tikzcd} \Omega_1 \arrow[r, "\varphi_1^{-1}"] \arrow[d, "\iota"'] & U_1 \arrow[d, "F|_{U_1}"] \\ \Theta_1 \arrow[r, "\psi_1^{-1}"'] & V_1 . \end{tikzcd}$$

If this is true, we are done, as U_1 and V_1 will be the answer.

8. Compute:

$$\psi_1^{-1} \circ \iota = \psi^{-1} \circ G \circ \iota = \psi^{-1} \circ \bar{F} = F \circ \varphi_1.$$

This ends the proof of the theorem. ☐

4.2 Submanifolds

Certain immersions, like $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ are special.

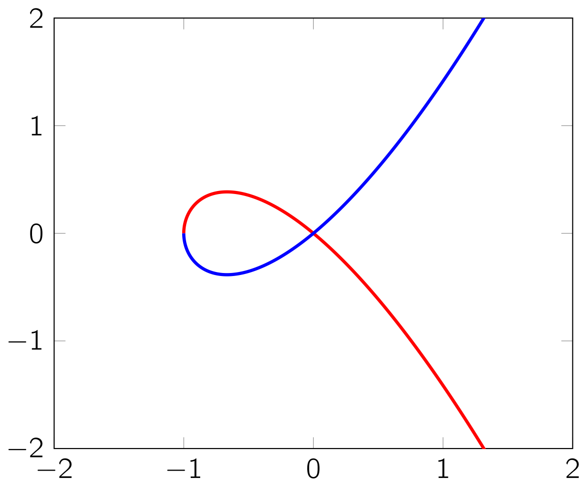
Definition 4.8. A smooth map $F : S \to M$ is an *embedding* if it is an immersion and induces a homeomorphism $S \cong F(S)$ (we put induced topology on $F(S)$).

Example 4.9. What is not an embedding?

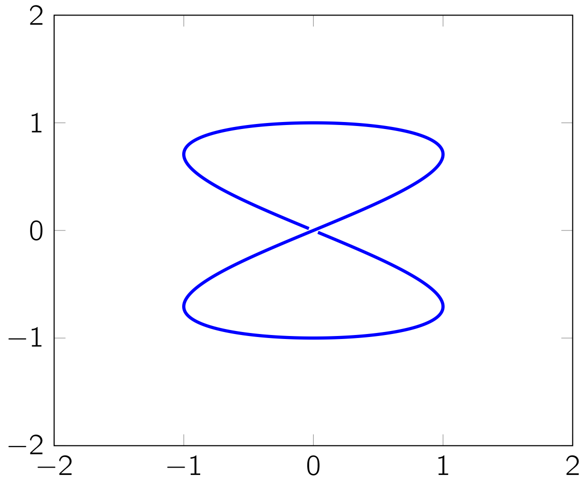
1. Consider a map $F : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$. Both functions are smooth and the differential is

$$dF(t) = {}^t(2t, 3t^2 - 1) \neq 0 \text{ for all } t \in \mathbb{R}.$$

However it has a self-intersection as it parametrieses the curve $y^2 = x^2(x + 1)$.



2. Consider a map $]-\pi/2, 3\pi/2[\to \mathbb{R}^2$, $t \mapsto (\sin 2t, \cos t)$. Its differential is again never zero but the image is the same as for its extension $[-\pi/2, 3\pi/2] \to \mathbb{R}^2$. There is a particular problem with the topology of the image around $(0, 0)$:



Embedded submanifolds

Definition 4.10. Let M be a smooth n-manifold and S a subset and $k \leq n$. We say that S is a (embedded) *k-submanifold* if for each point $x \in S$

1. There exists a chart $\varphi : (U, C_U^\infty) \xrightarrow{\sim} (\Omega, C_\Omega^\infty)$, such that
2. one has $\varphi(S \cap U) = \Omega \cap \mathbb{R}^k$, where we view $\mathbb{R}^k \subset \mathbb{R}^n$ via the inclusion $\iota : (x^1, ..., x^k) \mapsto (x^1, ..., x^k, 0, ..., 0)$.

   Such a chart (U, φ) is sometimes called a *slice chart*.

Lemma 4.11. *Let $F : K \to M$ be an immersion that induces a homeomorphism $K \xrightarrow{\sim} F(K)$. Then the set $S = F(K)$ is a k-submanifold, where $k = \dim K$.*

Proof. By canonical immersion theorem, for each $x \in K$ there is a diagram

$$\begin{tikzcd}
K \arrow[d, "F"'] & U \arrow[l, hook'] \arrow[r, "\varphi", "\sim"'] \arrow[d, "F"'] & \Omega \arrow[d, "\iota"] \\
M & V \arrow[l, hook'] \arrow[r, "\sim", "\psi"'] & \Theta.
\end{tikzcd}$$

As before, (U, φ) is a chart in K containing x, (V, ψ) is a chart in M and $\iota(x^1, ..., x^k) = (x^1, ..., x^k, 0, ..., 0)$.

1. Denote $\mathcal{U} := F(U) \subset S$. Note that for $y \in \mathcal{U}$, we have $\psi(y) \in \operatorname{im} \iota$, since $y = F(z)$ for some $z \in U$ and $\iota(\varphi(z)) = \psi(F(z))$.
2. The set $\mathcal{U}$ is contained in V by the diagram above. It is also open in S since $F : K \to S$ is a homeomorphism. This means that there exists $W \in \mathsf{Op}(M)$ such that $\mathcal{U} = W \cap S$.
3. Take $\mathcal{V} := W \cap V$. The pair $(\mathcal{V}, \psi|_{\mathcal{V}})$ satisfies the following properties:

   (a) It is a chart: we can write $\psi : \mathcal{V} \xrightarrow{\sim} \Theta_1 \subset \mathbb{R}^n$.

(b) The intersection $S \cap \mathcal{V}$ is equal to $\mathcal{U}$: $S \cap \mathcal{V} = S \cap (W \cap V) = \mathcal{U} \cap V = \mathcal{U}$.

(c) The chart map ψ sends $\mathcal{U}$ onto the subset

$$\{(x^1, ..., x^n) \in \Theta_1 \quad | \quad x^{k+1} = ... = x^n = 0\}$$

This finishes the proof for $(\mathcal{V}, \psi)$ can be found for any point $F(x)$ of S. □

Remark 4.12. The slice charts are not the ones obviously existing. Indeed, think of the spheres as embedded in $\mathbb{R}^n$!

Slice charts give manifold structure

Definition 4.13. Let $\mathcal{M}$ be a smooth manifold, and $A \subset M$ any subset. We say that $f : A \to \mathbb{R}$ *locally smoothly extends to* M if for each $x \in A$ there exists $U_x \in \mathsf{Op}(M)$, $x \in U_x$, and $g \in C^\infty_M(U_x)$ such that $g|_{A \cap U_x} = f|_{A \cap U_x}$.

Lemma 4.14. *Let $\iota : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ denote the first k coordinates inclusion. Let $f : \mathbb{R}^k \to \mathbb{R}$. Then the following are equivalent:*

1. *f is smooth on $\mathbb{R}^k$,*
2. *f locally smoothly extends to $\mathbb{R}^n$.*

Proof. If f is smooth then so is $\tilde{f} : (x_1, ...x_n) \mapsto f(x_1, ..., x_k)$. But $\tilde{f}|_{\mathbb{R}^k} = f$.

Conversely, if $V \subset \mathbb{R}^k$, $V = \mathbb{R}^k \cap U$ for some open $U \subset \mathbb{R}^n$, and $g : U \to \mathbb{R}$ is a smooth extension of $f|_V$, then $f|_V = g \circ \iota|_V$. We conclude since ι is smooth. □

Theorem 4.15. *(Slice criterion for submanifolds) Let M be a smooth n-manifold and S a k-submanifold in the sense of Definition 4.10. Then*

1. *There a manifold structure on S making it into a k-manifold.*
2. *This structure is uniquely determined by the requirements the topology on S is induced and that $i : (S, C^\infty_S) \hookrightarrow (M, C^\infty_M)$ is a smooth embedding.*
3. *Let $F : N \to M$ be any smooth map such that $\mathsf{im}\, F \subset S$. Then the map F read as $F : N \to S$ is also smooth.*

Proof. Short version: Put induced topology on S. Let $\mathcal{U} \in \mathsf{Op}(S)$. Define

$$C^\infty_S(\mathcal{U}) := \{f : \mathcal{U} \to \mathbb{R} \quad | \quad f \text{ locally smoothly extends to } M\}.$$

The verification below explains that with the help of slice charts and observations like that of Lemma 4.14, (S, C^∞_S) is a smooth manifold.

For F as in 3. we note that $f \in C^\infty_S(S)$ implies that $f \circ F$ can be written locally as $g \circ F$ for some $g \in C^\infty_M(U)$. The sheaf property helps to conclude that $F^*(f) : N \to \mathbb{R}$ is smooth.

Longer and tedious version, optional reading. Let us prove the first point.

1. Let $A \subset S$ any subset. We say that $f : A \to S$ locally extends to M if for each $x \in A$ there exists $U_x \in \mathsf{Op}(M)$ and $g \in C^\infty_M(U_x)$ such that $g|_{A \cap U_x} = f|_{A \cap U_x}$.

2. We consider S with induced topology. We will define the smooth structure sheaf by setting, for $\mathcal{U} \in \mathsf{Op}(S)$,

$$C^\infty_S(\mathcal{U}) := \{f : \mathcal{U} \to \mathbb{R} \quad | \quad f \text{ locally extends to } M\}.$$

3. Since the property of local extension is defined point-by-point, C^∞_S is a presheaf on S.

4. Let $\mathcal{U} = \cup \mathcal{U}_i$ in S, and consider $f : \mathcal{U} \to \mathbb{R}$ that has the local extension property when restricted to all $\mathcal{U}_i$. Since each $x \in \mathcal{U}$ belongs to some $\mathcal{U}_i$, we use the local extension there and conclude that f locally extends to M everywhere on $\mathcal{U}$.

5. Let $\mathcal{U} \in \mathsf{Op}\, S$ and $f_1, f_2 \in C^\infty_S(\mathcal{U})$. Then if g_1, g_2 extend f_1, f_2, we have that $\lambda g_1 + \mu g_2$ extends $\lambda f_1 + \mu f_2$, and $g_1 \cdot g_2$ extends $f_1 \cdot f_2$.

6. We conclude that (S, C^∞_S) is an $\mathbb{R}$-space.

7. Now, let $\varphi : U \xrightarrow{\sim} \Omega$ be a slice chart of M (that has nonempty intersection with S). Without loss of generality we suppose that Ω is an open ball centred at 0. Denote $\mathcal{U} = U \cap S$. Then the slice condition implies that $\Theta := \varphi(\mathcal{U})$ is a subset of $\mathbb{R}^k \cap \Omega$. The restrictions of continuous maps are continuous, so

$$\varphi|_{\mathcal{U}} : \mathcal{U} \leftrightarrows \Theta : \varphi^{-1}|_\Theta$$

is a homeomorphism. Denote $\psi := \varphi|_{\mathcal{U}}$.

8. Let $V \in \mathsf{Op}(\Theta)$ and $f : V \to \mathbb{R}$. Denote Assume that f is smooth on V. Defining $\tilde{f}(x^1, ..., x^n) = f(x^1, ..., x^k)$ trivially extends f to a smooth function $\tilde{f}$ defined on an open subset $\tilde{V}$ of Ω such that $\tilde{V} \cap \mathbb{R}^k = V$. Thus $f \circ \psi : \psi^{-1}(V) \to \mathbb{R}$ can be extended to $\tilde{f} \circ \varphi : \varphi^{-1}(\tilde{V}) \to \mathbb{R}$. This means that $\psi^*(f) \in C^\infty_S(\psi^{-1}(V))$.

9. Let $V \in \mathsf{Op}(\Theta)$ and $f : V \to \mathbb{R}$. Assume that $\psi^*(f) \in C^\infty_S(\psi^{-1}(V))$. Precisely, this means that there exists a cover $\cup \mathcal{V}_i$ of $\psi^{-1}(V)$ with each $\mathcal{V}_i = S \cap U_i$ such that $\psi^*(f)|_{\mathcal{V}_i}$ admits a smooth extension $g_i : U_i \to \mathbb{R}$. We can assume that each $U_i \subset U$, if not, intersect. Then the function f has the property that it is extendable (by a smooth $g_i \circ \varphi^{-1}$) in a neighbourhood of each point of V. This means that f is smooth.

The first point is proven. Using the slice charts we easily see that for $p \in S$, T_pS is identified with a k-subspace of T_pM (the map ι is inverse to π), meaning that the inclusion $S \hookrightarrow M$ is an immersion.

Uniqueness:

1. Assume that $\mathcal{A}$ is a different function sheaf that makes $(S, \mathcal{A})$ into a smooth manifold. The requirement that $j : S \hookrightarrow M$ is a smooth embedding tells us that $\dim S = k$. Let us prove that $\mathcal{A} = C^\infty_S$ as constructed above.

2. By canonical immersion theorem, for each $x \in S$ there is a diagram

$$\begin{tikzcd} S \arrow[d, "j"'] & U \arrow[l, hook'] \arrow[r, "\varphi", "\sim"'] \arrow[d, "j"'] & \Omega \arrow[d, "\iota"] \\ M & V \arrow[l, hook'] \arrow[r, "\sim", "\psi"'] & \Theta. \end{tikzcd}$$

As before, (U, φ) is a chart in S containing x, (V, ψ) is a chart in M and $\iota(x^1, ..., x^k) = (x^1, ..., x^k, 0, ..., 0)$.

3. Let $f \in \mathcal{A}(U)$ meaning that $f \circ \varphi^{-1}$ is smooth on Ω. As in Lemma 4.14, we can observe that there exists cover $\cap \Omega_i = \Omega$ such that $f \circ \varphi^{-1}$ is locally extensible by smooth $g_i : \Theta_i \to \mathbb{R}$, with $\Theta_i \in \mathsf{Op}\, \Theta$, $\Theta_i \cap \iota(\Omega) = \iota\Omega_i$.

4. Since $g_i \circ \psi \circ j = g_i \circ \iota \circ \varphi = f|_{\dots}$, we conclude that on $\varphi^{-1}(\Omega_i)$ f is locally extensible by $g_i \circ \psi$. This means that $\mathcal{A}(U) \subset C^\infty_S(U)$.

5. Same verification can be done for any subset of U. Sheaf property can then be used to conclude that $\mathcal{A}(U) \subset C^\infty_S(U)$ for any $U \in \mathsf{Op}(S)$.

6. On the other hand, let $f \in C^\infty_S(U)$. On some $V \subset U$, we can extend f to $g : \tilde{V} \to \mathbb{R}$, smooth on an open $\tilde{V} \subset M$. By assumption $j^*(g)$ is in $\mathcal{A}(j^{-1}\tilde{V})$. But $j^{-1}\tilde{V} = \tilde{V} \cap S = V$ and $j^*(g) = f|_V$. Thus $f|_V \in \mathcal{A}(V)$. We then use the sheaf property of $\mathcal{A}$ to conclude that $f \in \mathcal{A}(U)$.

The last point is proven the same way as in the short version. $\square$

Corollary 4.16. *Let $F : K \to M$ be a smooth embedding. Then $F(K)$ is a smooth manifold of dimension $\dim K$ and $F : K \to F(K)$ is a diffeomorphism.*

Proof. Theorem 4.15, Lemma 4.11 and Corollary 4.6 do most of the work. It is useful to comment on the pushforward maps. For $p \in K$, there is the following commutative diagram of pushforwards.

$$\begin{tikzcd} T_pK \arrow[rr] \arrow[dr] & & T_{F(p)}M \\ & T_{F(p)}F(K) \arrow[ur] & \end{tikzcd} .$$

The map $T_pK \to T_{F(p)}M$ is injective (of rank $\dim K$), and so is the map $T_{F(p)}F(K) \to T_{F(p)}M$. This can only happen if $T_pK \to T_{F(p)}F(K)$ is an isomorphism. $\square$

Submanifolds as level sets

The reason we bothered with such an uncomfortable notion is the following theorem, very powerful for constructing examples.

Theorem 4.17. *Let $F : M \to N$ be a smooth map. Let $q \in N$ be such that for each $p \in F^{-1}(q)$, the map $F_* : T_pM \to T_qN$ is surjective (in other words, F is a submersion on $F^{-1}(q)$). Then*

1. *The set $S = F^{-1}(q)$ is a k-submanifold, where $k = \dim M - \dim N$,*

2. *The tangent space T_pS at $p \in S$ is given by $\ker F_*$.*

This is extremely strong! For example, consider $F : \mathbb{R}^n \to \mathbb{R}$, $F(x_1, \dots, x_n) = \sum x_i^2$. Then the theorem gives us a smooth manifold structure on $\mathbb{S}^{n-1} = F^{-1}(1)$ without constructing any charts! We can further consider systems of equations to get more complex examples.

Proof. As usual, go for the canonical submersion theorem at $p \in S$:

$$\begin{tikzcd} M \arrow[d, "F"'] & U \arrow[l, hook'] \arrow[r, "\varphi", "\sim"'] \arrow[d, "F"'] & \Omega \arrow[d, "\pi"] \\ N & V \arrow[l, hook'] \arrow[r, "\sim", "\psi"'] & \Theta \end{tikzcd}$$

($p \in U$, $q \in V$). We can use translations to ensure that $\psi(q) = 0$.

Consider $U \cap S = U \cap F^{-1}(q)$. For $p' \in U$, the statement $p' \in U \cap S$ is equivalent to $F(p') = q$ which in turn is equivalent to $\psi(F(p')) = \psi(q) = 0$. Since $\psi \circ F = \pi \circ \varphi$, we have that $p' \in U \cap S$ is equivalent to $\varphi(p') \in \pi^{-1}(0)$, or in other words, that

$$\varphi(U \cap S) = \{(x^1, ..., x^m) \in \Omega \quad | \quad x^1 = ... = x^n = 0\}.$$

Thus U works as a slice chart for S (modulo reshuffling the coordinates), and Theorem 4.15 implies the first point.

Let $j : S \to M$ denote the inclusion (a smooth embedding by Theorem 4.15). Note that $F \circ j$ is a constant map valued at q. This means that for any $X \in T_pS$, we have $(F \circ j)_* X(f) = X(f(q)) = 0$. Thus $F_* \circ j_* = 0$, $\operatorname{im} j_* \subset \ker F_*$. The map j_* is injective and

$$\dim \operatorname{im} j_* = \dim \ker F_* = m - n = \dim M - \dim N$$

so we get the result. □

4.3 Applications

The toolkit that we developed took a lot of work, but now we can do all sorts of verifications.

Example 4.18. Recall that $\mathsf{GL}_n(\mathbb{R})$ is an open set of $\mathbb{R}^{n^2}$, and hence is an open submanifold. The function $\det : \mathsf{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ is known to be smooth (it can be written as a polynomial of the matrix coefficients). What about its differential?

If $n = 2$, we have

$$M = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \ \det M = x_1x_4 - x_3x_2$$

and so $d\det(M) = dx_1 \cdot x_4 + x_1 \cdot dx_4 - dx_3 \cdot x_2 - x_3 \cdot dx_2$. Here dx_i denote the linear forms in $\mathbb{R}^4$ dual to the canonical basis of $\mathbb{R}^4$, $dx_i(e_j) = \delta_{ij}$. In other words

$$d\det(M)(v_1, v_2, v_3, v_4) = v_1x_4 + x_1v_4 - v_2x_3 - x_2v_3.$$

It might be useful to mention that there is a result, called Jacobi's formula:

$$d\det M = \operatorname{tr}(adj(M) \cdot dM), \quad dM = \begin{pmatrix} dx_1 & dx_2 \\ dx_3 & dx_4 \end{pmatrix}$$

and $adj(M)$ is the adjunct matrix. This result is valid for all n.

Let us verify it for $n = 2$. For $M = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, we have $adj(M) = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$ and hence

$$adj(M) \cdot dM = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} \cdot \begin{pmatrix} dx_1 & dx_2 \\ dx_3 & dx_4 \end{pmatrix}$$

$$= \begin{pmatrix} x_4 \cdot dx_1 - x_2 \cdot dx_3 & \dots \\ \dots & -x_3 \cdot dx_2 + x_1 \cdot dx_4 \end{pmatrix},$$

and so its trace is $x_4 \cdot dx_1 - x_2 \cdot dx_3 - x_3 \cdot dx_2 + x_1 \cdot dx_4 = d\det(M)$.

In any case, if $d\det(M) = 0$, this can only happen if $x_1 = x_2 = x_3 = x_4 = 0$. This clearly cannot happen on $\mathsf{SL}_2(\mathbb{R}) = \det^{-1}(1)$.

Thus det is a submersion when restricted to $\mathsf{SL}_2(\mathbb{R})$, and so $\mathsf{SL}_2(\mathbb{R})$ is a $2^2 - 1$-submanifold of $\mathsf{GL}_2(\mathbb{R})$, and the uniqueness of the smooth structure (Theorem 4.15) means that the level set smooth structure of Theorem 4.17 coincides with the smooth structure constructed by hand in TD.

The Jacobi formula allows to make the same observations for arbitrary n, meaning that $\mathsf{SL}_n(\mathbb{R})$ is a (closed) smooth $n^2 - 1$-submanifold of $\mathsf{GL}_n(\mathbb{R})$.

Example 4.19. The subgroup $\mathsf{O}_2(\mathbb{R}) \subset \mathsf{GL}_2(\mathbb{R})$ consists of all matrices M such that ${}^tMM = I_2$. If we write $M = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ as before, then the orthogonality equation becomes:

$$\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_3^2 & x_1x_2 + x_3x_4 \\ x_1x_2 + x_3x_4 & x_2^2 + x_4^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $\mathsf{O}_2(\mathbb{R}) = F^{-1}(0)$, where $F : \mathsf{Mat}_2(\mathbb{R}) \to \mathbb{R}^3$ works as

$$F\left(\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}\right) = (x_1^2 + x_3^2 - 1, x_2^2 + x_4^2 - 1, x_1x_2 + x_3x_4).$$

As we now from last year, a matrix $M \in \mathsf{O}_2(\mathbb{R})$ is of the form $M = \begin{pmatrix} x_1 & x_2 \\ \mp x_2 & \pm x_1 \end{pmatrix}$. Compute the Jacobi matrix of F for such matrices:

$$J(F)(M) = \begin{pmatrix} 2x_1 & 0 & \mp 2x_2 & 0 \\ 0 & 2x_2 & 0 & \pm 2x_1 \\ x_2 & x_1 & \pm x_1 & \mp x_2 \end{pmatrix}.$$

Since $x_1^2 + x_2^2 = 1$, we see that the rank of $J(F)(M)$ is 3. Thus F is a submersion on $\mathsf{O}_2(\mathbb{R})$ meaning that the latter is a smooth submanifold of dimension $4 - 3 = 1$. It has two connected compoonents, one of them being $\mathsf{SO}_2(\mathbb{R})$. This is an open (and closed) subset of $\mathsf{O}_2(\mathbb{R})$ and hence is also a smooth 1-manifold.

Of course, we expect $\mathsf{SO}_2(\mathbb{R})$ to be diffeomorphic to $\mathbb{S}^1$ as a smooth manifold. Let us make it precise. Consider the map $D : \mathbb{R}^2 \to \mathsf{Mat}_2(\mathbb{R}) \cong \mathbb{R}^4$, defined as

$$D_0(x, y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

This map is obviously smooth of constant rank 2. Its restriction to $i : \mathbb{S}^1 \subset \mathbb{R}^2$ is the map $D = D_0 \circ i$, smooth of constant rank 1 (an immersion). It maps bijectively onto $\mathsf{SO}_2(\mathbb{R})$ as we know from MAA201. For $p \in \mathbb{S}^1$, in the diagram

$$\begin{tikzcd} T_p\mathbb{S}^1 \arrow[rr] \arrow[dr] & & T_{F(p)}\,\mathsf{Mat}_2(\mathbb{R}) \\ & T_{D(p)}\,\mathsf{SO}_2 \arrow[ur] & \end{tikzcd} .$$

we have that both $T_p\mathbb{S}^1 \to T_{F(p)}\,\mathsf{Mat}_2(\mathbb{R})$ and $T_{D(p)}\,\mathsf{SO}_2 \to T_{F(p)}\,\mathsf{Mat}_2(\mathbb{R})$ are of constant rank 1, so the map $D_* : T_pS_1 \to T_{D(p)}\,\mathsf{SO}_2$ cannot be zero. Thus D is a bijection and D_* is an isomorphism at each point, so D is a diffeomorphism.

Example 4.20. (All tori are the same) The product $\mathbb{S}^1 \times \mathbb{S}^1$ can be viewed as a smooth submanifold in $\mathbb{R}^2 \times \mathbb{R}^2$ defined as $F^{-1}(0,0)$ for

$$F : \mathbb{R}^4 \to \mathbb{R}^2 \quad F(x,y,z,t) = (x^2 + y^2 - 1, z^2 + t^2 - 1).$$

Since F is a submersion on $F^{-1}(0,0)$, the set $\mathbb{S}^1 \times \mathbb{S}^1$ gets the natural submanifold structure. It is compatible with all the usual maps and gives in fact the same product smooth structure as studied in TD.

We consider the map

$$P : \mathbb{R}^2 \to \mathbb{R}^4, \quad P(x,y) = (\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y)).$$

It is evidently smooth with image $\mathbb{S}^1 \times \mathbb{S}^1$, so we can view it as a smooth map of manifolds $\mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$. It also respects the torus equivalence relation. The induced map

$$\tilde{P} : \mathbb{T} \to \mathbb{S}^1 \times \mathbb{S}^1, \quad P = \tilde{P} \circ q$$

is also smooth since $q^*(\tilde{P}^*)(f) = P^*(f)$, so $f \in C^\infty(S^1 \times S^1)$ implies $(\tilde{P}^*)(f) \in C^\infty(\mathbb{T})$. The map $\tilde{P}$ remains a bijection. On the level of tangent spaces, the following commutes

$$\begin{tikzcd} T_p\mathbb{R}^2 \arrow[r] \arrow[d, "\sim"'] & T_{P(p)}\mathbb{R}^4 \\ T_{q(p)}\mathbb{T} \arrow[r, "\tilde{P}_*"'] & T_{P(p)}\mathbb{S}^1 \times \mathbb{S}^1 \arrow[u] \end{tikzcd} .$$

The maps $T_p\mathbb{R}^2 \to T_{P(p)}\mathbb{R}^4$ and $T_{P(p)}\mathbb{S}^1 \times \mathbb{S}^1 \to T_{P(p)}\mathbb{R}^4$ are of constant rank 2. For this reason $\tilde{P}_* : T_{q(p)}\mathbb{T} \to T_{P(p)}\mathbb{S}^1 \times \mathbb{S}^1$ can be only of rank 2, and so again $\tilde{P}$ is bijective of maximal rank, hence a diffeomorphism.

Exact same checks are possible for

$$G(x,y) = ((2 + \cos 2\pi x)\cos 2\pi y, (2 + \cos 2\pi x)\sin 2\pi y, \sin 2\pi x)$$

to establish a diffeomorphism between $\mathbb{T}$ and $G(\mathbb{R}^2)$. We leave it as an exercise.

5 Vector fields

We saw the expressions $\sum V^i \frac{\partial}{\partial x^i}|_p$ corresponding to p-derivations of $C^\infty(\Omega)$.

It also makes sense to consider $p \mapsto \sum V^i(p) \frac{\partial}{\partial x^i}\big|_p$, where all V^i are now smooth functions on Ω. Such a formula can take a smooth function f and produce a new function, $f \mapsto X(f)$, $X(f)(p) = X^i \partial_i f(p)$, again smooth. The operator X is called a vector field.

We would like to properly define it in the context of manifolds. One approach is to treat X as a family $p \mapsto X_p \in T_pM$ that "varies smoothly" (need to formalise what it means). This is advantageous because then it is easy how to restrict vector fields.

Another approach would be to define X as a map $C^\infty(M) \to C^\infty(M)$ that is $\mathbb{R}$-linear and satisfies Leibniz rule. This is very algebraic, but it is not clear how to restrict such expressions.

We will do both, and compare.

5.1 Vector fields as families

Definition 5.1. Let (M, C^∞_M) be a smooth manifold. A *(smooth) vector field* X on $U \in \mathsf{Op}\, M$ is an assignment, to each $p \in U$, of $X_p \in T_pM$, such that the following is satisfied. For each $V \in \mathsf{Op}\, U$ and $f \in C^\infty_M(V)$, the function

$$X(f) : p \mapsto X_p(f)$$

is smooth: $X(f) \in C^\infty_M(V)$. We denote the set of all vector fields on U as $\mathcal{T}_M(U)$.

We will slightly redefine the notion of assignment $p \mapsto X_p$ in a couple of moments.

Lemma 5.2. *The set $\mathcal{T}_M(U)$ has a natural vector space structure given by*

$$(\lambda X + \mu Y)_p := \lambda X_p + \mu Y_p.$$

Moreover, if $X \in \mathcal{T}_M(U)$ amd $g \in C^\infty_M(U)$, then $g \cdot X$ defined as $(gX)_p = g(p)X_p$ is again in $\mathcal{T}_M(U)$. In algebraic terms, this means that $\mathcal{T}_M(U)$ is a $C^\infty_M(U)$-module.

Proof. Standard check using the fact that linear combinations and products of smooth functions are smooth. $\square$

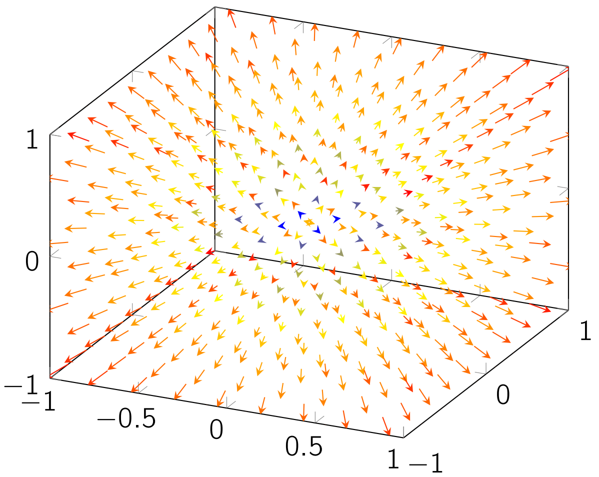
Example 5.3. Take $\Omega \in \mathsf{Op}\, \mathbb{R}^n$. Then for each $p \in \Omega$, there is a basis of $T_p\Omega$ given by $\{\partial/\partial x^i|_p\}$. The family

$$\partial_i : p \mapsto \frac{\partial}{\partial x^i}\bigg|_p$$

is a smooth vector field: indeed, for any $U \subset \Omega$ and $f \in C^\infty(U)$ the function $p \mapsto \partial_i f(p)$ is smooth. Previous lemma guarantees us that any sum $X = \sum X^i \partial_i$ with X^i smooth on Ω provides a vector field.

Conversely, let X be a smooth vector field on Ω as defined above. Then we can still write $X = \sum X^i \partial_i$ (because $\partial_i|_p$ is basis at each p), but for the moment we do not know if $p \mapsto X^i(p)$ is smooth.

This is almost immediate: if we denote $x^i : p \mapsto x^i(p)$ the i-th coordinate function, then $X(x^i) = X^i$, and it has to be smooth on Ω by definition.



$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

Tangent sheaf

Our definition of vector fields is adapted to restrictions: if $X \in \mathcal{T}_M(U)$ and $V \subset U$, then simply restricting to points in V defines $X|_V \in \mathcal{T}_M(V)$. This suggests that $\mathcal{T}_M = \{\mathcal{T}_M(U)\}_U$ could be a (pre)sheaf. To make it precise, denote

$$TM := \bigsqcup_{p \in M} T_pM = \{(p, v) \quad | \quad v \in T_pM\}.$$

This set is called the *tangent bundle* of M. It can be made into a smooth manifold, but we ignore this for now.

There is a function $\pi : TM \to M$ that sends (p, v) to p. Each smooth vector field can be viewed as a *section* of π. That is, if $X \in \mathcal{T}_M(U)$ then the map that we also denote

$$X : U \to TM, \quad p \mapsto (p, X_p)$$

evidently satisfies $\pi \circ X = \mathrm{id}_U$. In fact, defining a vector field X as a section of π over U is a more rigorous way to give meaning to "a collection of tangent vectors X_p for each $p \in U$". We thus can view $\mathcal{T}_M(U)$ as a subset of $\mathcal{F}(U, TM)$.

Proposition 5.4. *The collection $U \mapsto \mathcal{T}_M(U)$ forms a sheaf $\mathcal{T}_M$ of functions to TM, called the* tangent sheaf *of M.*

Proof. If $X \in \mathcal{T}_M(U)$, this means that as a map $X : U \to TM$ it satisfies $\pi \circ X = \mathrm{id}_U$. For $V \subset U$, we then immediately have $\pi \circ X|_V = \mathrm{id}_V$, meaning that $X(p)$ can be still written as (p, X_p).

Moreover the function $X(f)(p) := X_p(f)$ is smooth for any smooth function f defined on an open subset of V (and hence, of U). Thus the restriction $X|_V$ remains a smooth vector field, meaning that $\mathcal{T}_M$ is a presheaf.

Assume now that $U = \cup_i U_i$. Let $X : U \to TM$ be any map, and we know that $X|_{U_i} \in \mathcal{T}_M(U_i)$. This already implies that for each $p \in U$, the composition $\pi \circ X = \mathrm{id}_U$ as this is a point-by-point equality that can be verified on each U_i. Hence X can be written as a map $p \mapsto (p, X_p)$.

Now, if $V \subset U$ and $f \in C^\infty(V)$, then $X(f)(p) = X_p(f)$ satisfies the property of being a smooth function on each $U_i \cap V$. Since smooth functions form a sheaf, it means that $X(f)$ is smooth on $V = \cup V \cap U_i$. Thus X is a smooth vector field on U. $\square$

We apply the sheaf property of $\mathcal{T}_M$ to formulate the expected criterion of smoothness in terms of local coordinates. Let (V, φ) be a smooth chart. Denote, for $W \in \mathsf{Op}\, V$,

$$e_i^\varphi : \quad f \in C^\infty_M(W) \mapsto \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \circ \varphi.$$

Thanks to Example 5.3 this assignment gives a smooth function on W. This is a collection of tangent vectors: $(e_i^\varphi)_p$ coincides with $\partial_i|_p$ in the sense of Notation 3.33:

$$e_i^\varphi(f)(p) = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) = \varphi_*^{-1} \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} f.$$

Corollary 5.5. *Let $X : p \mapsto X_p$ be a family of tangent vectors for $p \in U$. The following are equivalent:*

1. *$X \in \mathcal{T}_M(U)$,*
2. *For each point $p \in U$ there exists a chart (V, φ), $p \in V \subset U$, such that*

$$X = \sum X^i e_i^\varphi,$$

where each $X^i \in C^\infty_M(V)$ and e_i^φ are defined as above.

Proof. Of course, for each $p \in U$ there is a chart V that contains it and is contained in U: often we can find a bigger chart in M and then intersect it with U. For a chart (V, φ), write $\varphi(p) = (x^1_\varphi(p), ..., x^n_\varphi(p))$ so that each $x^i_\varphi \in C^\infty_M(V)$: evidently, $x^i_\varphi \circ \varphi^{-1}$ is simply the i-th coordinate projection on $\Omega = \varphi(V)$.

1. $\Rightarrow$ 2. Let $X \in \mathcal{T}_M(U)$. Let (V, φ) be any chart around p inside U. Then $e_1^\varphi, \dots, e_n^\varphi$ belong to $\mathcal{T}_M(V)$ due to our previous considerations, and at each point $q \in V$, the q-derivations $(e_1^\varphi)_q, \dots, (e_n^\varphi)_q$ form a basis of $T_q M$.

   Thus we can write $X = \sum X^i e_i^\varphi$. The functions $X^i : V \to \mathbb{R}$ are smooth: it suffices to compute $X(x^i)$ on the smooth coordinate functions $x_\varphi^i : V \to \mathbb{R}$.

2. $\Rightarrow$ 1. Let X be a family of tangent vectors at points of U. Let (V, φ) be the chart around p inside U for which $X = \sum X^i e_i^\varphi$. Since e_i^φ are in $\mathcal{T}_M(V)$ and X^i are smooth on V, Lemma 5.2 implies that $\sum X^i e_i^\varphi$ is in $\mathcal{T}_M(V)$. We then cover U by all such V and use the sheaf property of $\mathcal{T}_M$ to conclude that $X \in \mathcal{T}_M(U)$. $\square$

Remark 5.6. Note that we (somewhat subtly, as usual with the sheaf approach) avoided the question of coordinate transformations. In practice, in order to construct $X \in \mathcal{T}_M(M)$, one often covers M by charts U_i, constructs $X|_{U_i} \in \mathcal{T}_M(U_i)$ and then verifies that $(X|_{U_i})|_{U_j} = (X|_{U_j})|_{U_i}$ on all $U_i \cap U_j$. This usually requires to perform coordinate transformations by the means of functions $\psi \circ \varphi^{-1}$.

What is it with the TM?

In our interpretation of vector fields as sheaves of functions, we passed by $TM = \sqcup_p T_p M$, a set. In fact, the following result is true:

Proposition 5.7. *There exists unique structure on TM making it into a smooth manifold such that*

1. *The projection $\pi : TM \to M$ is a smooth map,*
2. *$X \in \mathcal{T}_M(U)$ if and only if the section $M \to TM$, $p \mapsto (p, X_p)$ is a smooth map.*

This smooth manifold is called the tangent bundle of M.

Due to time limits, this will be an exercise for TD (that will or will not be explained). All books on differential geometry explain this, and of course, various manifolds appear in practice as tangent bundles to some other manifolds.

From an algebraic perspective, the vector fields sheaf $\mathcal{T}_M$ is however an object tied more fundamentally to C_M^∞, something that we are about to explain.

5.2 Vector fields as derivations

Definition 5.8. A *derivation* of an $\mathbb{R}$-algebra A is an $\mathbb{R}$-linear map $D : A \to A$ that satisfies the Leibniz rule: $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$.

We denote $\mathsf{Der}(A)$ the set of derivations of A. This set is naturally an $\mathbb{R}$-vector space with operations defined value-wise: $(\lambda D + \mu D')(a) := \lambda D(a) + \mu D'(a)$.

Let $X \in \mathcal{T}_M(U)$. Then X naturally defines a derivation of $C_M^\infty(U)$. Indeed, $f \mapsto X(f)$ is $\mathbb{R}$-linear in f as this is true for all $p \mapsto X_p(f)$. Moreover, for every p,

$$X(f \cdot g)(p) = X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)$$

and thus we can write $X(f \cdot g) X(f) \cdot g + f \cdot X(g)$. There is thus a natural map $\mathcal{T}_M(U) \to \mathsf{Der}(C_M^\infty(U))$.

Proposition 5.9. *The natural map* $\mathcal{T}_M(U) \to \mathsf{Der}(C^\infty_M(U))$ *constructed above is an isomorphism.*

The vector fields seem so similar to derivations it is hard to understand what is to prove here! The subtlety is that derivations only know about functions on U and not on its subsets.

Proof.

1. Without loss of generality, we can put $U = M$. For the time being, let us introduce some notation to distinguish between vector fields and derivations. If X is a vector field then $X_D : C^\infty(M) \to C^\infty(M)$ is its associated derivation, given by action of X.

2. For each $p \in M$ there is a natural map $p_* : \mathsf{Der}(C^\infty_M) \to \mathsf{Der}_p(C^\infty_M, \mathbb{R})$ that acts as follows.
$$D \mapsto p_*D, \quad p_*D(f) := D(f)(p).$$
This map sends derivations to derivations since functional Leibniz rule gives rise to pointwise Leibniz rule. Note that $D = 0$ iff for all $p \in M$, $p_*D = 0$.

3. Let X such that $X_D = 0$. However, $p_*X_D(f) = X(f)(p) \equiv X_p(f)$ and so $p_*X_D = X_p$. Thus $X_p = 0$ for each p, meaning that $X = 0$. Our map $\mathcal{T}_M(M) \to \mathsf{Der}(C^\infty_M(M))$ is thus injective.

4. Let $D \in \mathsf{Der}(C^\infty(M))$. We want to construct a vector field that produces it. For this, we consider the collection $X_p = p_*D$. We need to test if for each $f \in C^\infty(V)$, the function $p \mapsto X_p(f)$ remains smooth on V.

5. For this, we use two points from the proof of Proposition 3.29:
   (a) For each $p \in V$, there exists a global function $g \in C^\infty(M)$ that coincides with f on some $W \subset V$ that contains p,
   (b) If $f|_W = g|_W$ then for each $q \in W$ and each $Y \in T_qW$, we have $Y(f) = Y(g)$.
   (c) Result of these points being that $T_qW \cong T_qV \cong T_qM$ canonically.

   Consider the function $q \mapsto X_q(f)$ for $q \in W$. Because of the above remarks, it is the same as the function $q \mapsto X_q(g)$ for $q \in W$. However, the latter is simply the restriction $D(g)|_W \in C^\infty(W)$. We cover V by varying p and get that $p \mapsto X_p(f)$ is smooth on V.

6. The proof is over since $X_D(f)(p) = X_p(f) = p_*D(f) = D(f)(p)$ for all $p \in M$. $\square$

This proposition can be read the other way around. We could have simply defined the vector fields as derivations. Then we verify that $U \mapsto \mathsf{Der}(C^\infty_M(U))$ is in fact a sheaf (in a more abstract sense than a sheaf of functions). This way one does not even need to introduce tangent spaces! In fact, they are subtly hidden inside the structure of the sheaf of vector fields.

Example 5.10. Recall the smooth map $F : \mathbb{R} \to \mathbb{S}^1$, $\alpha \mapsto (\cos(2\pi\alpha), \sin(2\pi\alpha))$. There is a global vector field $\frac{d}{d\alpha} \in \mathcal{T}_{\mathbb{R}}(\mathbb{R})$.

The map $F^* : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{R})$ is a bijection onto the $\mathbb{R}$-subalgebra $C^\infty_{per}(\mathbb{R})$ of functions f such that $f(\alpha + n) = f(\alpha)$. This can be proven in a way similar to the torus case: we can indeed verify that the quotient smooth structure on $\mathbb{R}/\mathbb{Z}$ is diffeomorphic to $\mathbb{S}^1$.

Using this argument one can check that $(F^*)^{-1} : C^\infty_{per}(\mathbb{R}) \to C^\infty(\mathbb{S}^1)$ acts as follows:

$$(F^*)^{-1}(f)(p) = f(\alpha) \text{ where } \alpha \text{ satisfies } p = (\cos 2\pi\alpha, \sin 2\pi\alpha).$$

Let us construct $X \in \mathcal{T}^1_{\mathbb{S}}(\mathbb{S}^1)$. Note that for $f \in C^\infty(\mathbb{S}^1)$, the derivative $d(f \circ F)/d\alpha$ also belongs to $C^\infty_{per}(\mathbb{R})$. Let us write

$$X(f) := (F^*)^{-1}\left(\frac{d(F^*(f))}{d\alpha}\right).$$

This is smooth on $\mathbb{S}^1$. Moreover,

$$d_\alpha(F^*(f \cdot g)) = F^*(f) \cdot d_\alpha F^*(g) + d_\alpha F^*(f) \cdot F^*(g).$$

Since F^* preserves the $\mathbb{R}$-algebra structure, so does $(F^*)^{-1} : C^\infty_{per}(\mathbb{R}) \to C^\infty(\mathbb{S}^1)$. We get that

$$(F^*)^{-1} d_\alpha(F^*(f \cdot g)) = f \cdot (F^*)^{-1}(d_\alpha F^*(g)) + (F^*)^{-1}(d_\alpha F^*(f)) \cdot g$$

so X is indeed a smooth vector field on $\mathbb{S}^1$.

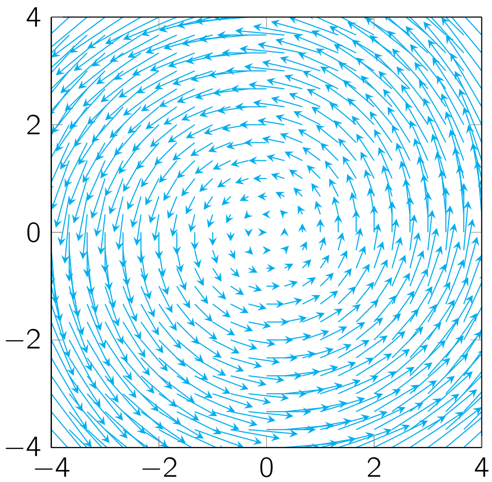
If $p \in \mathbb{S}^1$ is such that $p = F(\alpha_0) = (\cos(2\pi\alpha_0), \sin(2\pi\alpha_0))$, then

$$X_p = F_*\left(\left.\frac{d}{d\alpha}\right|_{\alpha_0}\right).$$

If we denote by $i : \mathbb{S}^1 \to \mathbb{R}^2$ the inclusion map, then we can compute slightly further to see that

$$i_* X_p = 2\pi x \left.\frac{\partial}{\partial y}\right|_p - 2\pi y \left.\frac{\partial}{\partial x}\right|_p.$$

This is orthogonal to the radial vector and runs around counterclockwise with constant speed (as expected).



$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

5.3 Lie bracket

The derivation perspective has some interesting consequences.

Lemma 5.11. *Let D_1, D_2 be two derivations of an $\mathbb{R}$-algebra A. Then*

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

(the commutator) is again a derivation.

Proof. The $\mathbb{R}$-linearity being obvious (compositions of linear maps), for $a, b \in A$, we compute

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(ab)) - D_2(D_1(ab)) \\
&= D_1(aD_2(b) + D_2(a)b) - D_2(aD_1(b) + D_1(a)b) \\
&= D_1(a)D_2(b) + aD_1(D_2(b)) + D_1(D_2(a))b + D_2(a)D_1(b) \\
&\quad - D_2(a)D_1(b) - aD_2(D_1(b)) - D_2(D_1(a))b - D_1(a)D_2(b) \\
&= (D_1(D_2(a)) - D_2(D_1(a))\, b + a\,(D_1(D_2(b)) - D_2(D_1(b))) \\
&= ([D_1, D_2](a))\,(b) + a\,([D_1, D_2](b))\,.
\end{aligned}$$

The bracket thus satisfies the Leibniz rule. □

Corollary 5.12. *Let (M, C^∞_M) be a smooth manifold, $U \in \mathsf{Op}\, M$ and $X, Y \in \mathcal{T}_M(U)$. Then the assignment*

$$f \mapsto X(Y(f)) - Y(X(f))$$

defines a vector field $[X, Y] \in \mathcal{T}_M(U)$ and called the Lie bracket of X, Y. □

Lemma 5.13. *Let $X, Y \in \mathcal{T}_M(U)$ and $V \subset U$ open. Then for $f \in C^\infty_M(V)$,*

$$[X, Y]|_V(f) = [X|_V, Y|_V](f).$$

Proof. Extend f by $\tilde{f} \in C^\infty_M(U)$ that agrees with f on some $W \ni p$. Then

$$\begin{aligned}
[X, Y]|_V(f)(p) &= [X, Y](\tilde{f})(p) = X(Y(\tilde{f}))(p) - Y(X(\tilde{f}))(p) \\
&= X(\widetilde{Y|_V(f)})(p) - Y(\widetilde{X|_V(f)})(p) \\
&= X|_V(Y|_V(f))(p) - Y|_V(X|_V(f))(p).
\end{aligned}$$

In this computation, we have used the following observation: if $\tilde{f}$ is an U-extension of f to that agrees with f on W, then $X(\tilde{f}) : p \mapsto X_p(\tilde{f})$ is an U-extension of $X|_V(f) : p \mapsto X_p(f)$ that agrees with $X|_V(f)$ on W. □

Example 5.14. At a point $p \in U$, we have $[X, Y](f)(p) = X_p(Y(f)) - Y_p(X(f))$; please note that writing $X_p(Y_p(f)) - Y_p(X_p(f))$ is not the same! If we interpret $X_p(f) = X(f)(p)$ as a constant function, then the latter expression of the previous sentence is always zero. However, even on $\mathbb{R}^n$ this is not always the case.

Consider two vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^i \frac{\partial}{\partial x^i}$ on $\Omega \in \mathsf{Op}\, \mathbb{R}^n$. Then a little exercise shows that the Lie bracket is

$$[X, Y] = \sum_{i,j} \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Lemma 5.15. *The Lie bracket* $[,] : \mathcal{T}_M(U) \times \mathcal{T}_M(U) \to \mathcal{T}_M(U)$ *is a bilinear antisymmetric map. For* $X, Y, Z \in \mathcal{T}_M(U)$*, the Lie bracket satisfies the Jacobi identity:*

$$[[X, Y], Z] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

Proof. Derivations are in particular linear maps. The same facts are true for commutators of linear maps. □

Lie algebra of vector fields

Definition 5.16. A *Lie algebra* $\mathfrak{g}$ is an $\mathbb{R}$-vector space together with a bilinear antisymmetric map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called Lie bracket) that satisfies the Jacobi identity: for each $a, b, c \in \mathfrak{g}$, we have

$$[[a, b], c] + [c, [a, b]] + [b, [c, a]] = 0.$$

Given two lie algebras $(\mathfrak{g}, [\,,\,]_\mathfrak{g})$, $(\mathfrak{h}, [\,,\,]_\mathfrak{h})$ A *Lie algebra homomorphism* is an $\mathbb{R}$-linear map $f : \mathfrak{g} \to \mathfrak{h}$ that preserves the brackets: for any $a, b \in \mathfrak{g}$, we have

$$f([a, b]_\mathfrak{g}) = [f(a), f(b)]_\mathfrak{h}.$$

All our observations can now be summarised as follows.

Corollary 5.17. *For each* $U \in \mathsf{Op}\, M$*, the pair* $(\mathcal{T}_M(U), [\,,\,])$ *is a Lie algebra, and for each* $V \in \mathsf{Op}\, U$*, the restriction maps* $\mathcal{T}_M(U) \to \mathcal{T}_M(V)$ *are Lie algebra homomorphisms.* □

This, together with the sheaf property of $\mathcal{T}_M$, allows to compute Lie brackets locally. It would be nice to understand one more thing: can we actually compute them by passing to charts, without worrying that the chart diffeomorphisms φ get in the way?

Pushforwards of vector fields

Contrary to tangent vectors, pushing forward vector fields is not obvious. For example, there is no canonical way to extend a vector field from $U \in \mathsf{Op}\, M$ to M.

Let $F : M \to N$ be a smooth map and consider $X \in \mathcal{T}_M(M) = \mathsf{Der}(C^\infty(M))$. We can consider something like $F^* \circ X : C^\infty(M) \to C^\infty(M) \to C^\infty(N)$, but it will not be a vector field on N:

$$\begin{tikzcd} C^\infty(M) \arrow[r, "X"] & C^\infty(M) \\ C^\infty(N) \arrow[u, "F^*"] \arrow[r, dashed, "?"'] & C^\infty(N). \arrow[u, "F^*"'] \end{tikzcd}$$

Definition 5.18. Let $F : M \to N$ be a smooth map, $X \in \mathcal{T}(M), Y \in \mathcal{T}(N)$. We say that X and Y are F*-related*, if for each $p \in M$, one has $F_*(p)X_p = Y_{F(p)}$, where $F_*(p) : T_pM \to T_{F(p)}N$ is the pushforward at p.

Example 5.19. We saw in Example 5.10 that there is a vector field on S^1 that is related to $x\partial_y - y\partial_x$ via the inclusion map.

If $F : M \to N$ is a diffeomorphism, then the situation simplifies enormously.

Proposition 5.20. *Let $F : M \xrightarrow{\sim} N$ be a diffeomorphism, with inverse G. For $X \in \mathcal{T}(M)$, define $F_*X := G^* \circ X \circ F^* : C^\infty(N) \to C^\infty(N)$. Then*

1. *F_*X is a vector field on N that is F-related to X.*
2. *For $X, Y \in \mathcal{T}(M)$, one has $F_*[X, Y] = [F_*X, F_*Y]$.*

Proof. To check 1., we do the derivation test. As usual, note that $F^*(fg) = F^*(f)F^*(g)$. Thus

$$X(F^*(fg)) = X(F^*(f)) \cdot F^*(g) + F^*(f) \cdot X(F^*(g)).$$

It remains to apply G^* (which also respects multiplication) and use the fact that $G^*F^* = (FG)^* = \mathsf{id}$:

$$G^*(X(F^*(fg))) = G^*(X(F^*(f))) \cdot g + f \cdot G^*(X(F^*(g))).$$

Thus F_*X is a vector field. Note that for $q = F(p)$,

$$G^*(X(F^*(f)))(q) = X(F^*(f))(p) = X_p(F^*(f)) = (F_*(p)X_p)(f)$$

thus the F-relation is in order.

To prove 2., simply note that

$$F_*X \circ F_*Y = (G^* \circ X \circ F^*) \circ (G^* \circ Y \circ F^*) = G^* \circ X \circ Y \circ F^*$$

and we can write the same for $F_*Y \circ F_*X$. Thus their difference is

$$F_*X \circ F_*Y - F_*Y \circ F_*X = G^* \circ (X \circ Y - Y \circ X) \circ F^*$$

and that proves the proposition. □

Consequently, to compute a Lie bracket $[X, Y]$, we can restrict it to a chart U, and then use the diffeomorphism $\varphi : U \xrightarrow{\sim} \Omega$ to do local coordinate computations.

The Lie bracket is quite robust even if F is not a diffeomorphism:

Proposition 5.21. *Let $F : M \to N$ be a smooth map. Assume that $X_1, X_2 \in \mathcal{T}(M)$ are F-related to $Y_1, Y_2 \in \mathcal{T}(N)$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Proof. Exercise. It is useful to first prove the following characterisation of F-relation: X is F-related to Y iff for all $V \in \mathsf{Op}\, N$ and $f \in C^\infty_N(V)$, one has $X(F^*(f)) = F^*(Y(f))$ (here we treat X, Y as vector fields in the sense of our original definition). □

Finally, the following is a nice exercise on slice charts:

Exercise 5.22. *Let M be a smooth manifold and S an embedded submanifold. Denote $i : S \to M$ the inclusion map. Let $X \in \mathcal{T}_M(M)$ be such that for all $p \in S$, one has $X_p \in i_*(T_pS) \subset T_pM$. Then there exists a vector field on S, denoted $X|_S$ that is i-related to X.*

The implication of these two statements may seem non-obvious: Lie brackets do not gain "extra directions" when passing from submanifolds to manifolds.

5.4 Integral curves

Let $I \subset \mathbb{R}$ be an open interval, a smooth 1-dimensional manifold. We can consider smooth maps $\gamma : I \to M$ to any other smooth manifold, and call them curves in M. We previously defined, for $t_0 \in I$,

$$\gamma'(t_0) := \gamma_*(t_0)\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_pM.$$

Definition 5.23. Let $X \in \mathcal{T}(M)$. A smooth map $\gamma : I \to M$ is called an *integral curve* of X

$$\gamma'(t) = X_{\gamma(t)} \quad \forall t \in I;$$

put differently, γ relates the standard vector field $d_t = d/dt$ on I to X.

Example 5.24. Let us decipher what it means for $M = \mathbb{R}^n$. In this case, a smooth map $\gamma : I \to \mathbb{R}^n$ can be written as $t \mapsto (\gamma^1(t), ..., \gamma^n(t))$. In this case

$$\gamma'(t) = \sum_i \frac{d\gamma^i}{dt}(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

and thus for a vector field $X = \sum X^i\partial_i$ on $\mathbb{R}^n$, the integral curve equation becomes

$$\frac{d\gamma^i}{dt}(t) = X^i(\gamma(t)), \quad i = 1, ..., n.$$

Example 5.25. Very concretely, let us take our old friend $X = x\partial_y - y\partial_x$ on $\mathbb{R}^2$. The equations for $\gamma : I \to \mathbb{R}^2$ become

$$\dot{\gamma}^1 = -\gamma^2, \quad \dot{\gamma}^2 = \gamma^1.$$

This is of course solvable by circles $\gamma(t) = (R\cos(t+\alpha), R\sin(t+\alpha))$, and we can put $I = \mathbb{R}$.

We are thus dealing with systems of ordinary differential equations (ODE) in the case of $\mathbb{R}^n$, or its open subsets. If we work with general manifolds, charts still allow such a translation.

Assume that $\gamma : I \to M$ has its image contained in a chart $\varphi : U \xrightarrow{\sim} \Omega$. For any $X \in \mathcal{T}_M(U)$, the equation $\gamma'(t) = X_{\gamma(t)}$ can be then translated in Ω. Simply push X forward using φ. Then we can write our equation as

$$(\varphi \circ \gamma)'(t) = \varphi_*\gamma'(t) = \varphi_*X_{\gamma(t)}.$$

To polish it slightly more, denote $\gamma_\varphi = \varphi \circ \gamma$, and $Y^\varphi = (\varphi^{-1})^* \circ X \circ \varphi^*$ the vector field on Ω obtained by taking pushforward of X along the diffeomorphism ψ. For a point $x \in \Omega$, one has $Y^\varphi_x(f) = X_{\varphi^{-1}(x)}(f \circ \varphi)$. The equation becomes

$$\gamma'_\varphi(t) = Y^\varphi_{\tilde{\gamma}(t)}.$$

It is easy to verify that given another chart structure ψ, the solutions of $\gamma'_\varphi = Y^\varphi$ are mapped to $\gamma'_\psi = Y^\psi$. We can then use the machinery of ODE to formulate results.

Remark 5.26. Recall from TD that given two smooth manifolds M, N, their product $M \times N$ inherits a smooth structure making it into a product in the sense of the universal property.

In TD, this structure was constructed by taking atlases: a chart of $M \times N$ is given by $(U \times V, \varphi \times \psi)$ where (U, φ) and (V, ψ) are charts of M and N respectively.

A pair of smooth maps $K \to M$, $K \to N$ induces unique smooth map $K \to M \times N$. In particular, fixing a point $p \in M$ produces a smooth map $N \overset{p\times \mathrm{id}_N}{\longrightarrow} M \times N$.

On the other hand, we can characterise what it means for $F : M \times N \to K$ to be smooth. Using chart diffeomorphism, it is sufficient to require that for each chart $(U \times V, \varphi \times \psi)$ of $M \times N$, the map

$$\varphi(U) \times \psi(V) \overset{\varphi^{-1}\times\psi^{-1}}{\longrightarrow} U \times V \subset M \times N \overset{F}{\longrightarrow} K$$

is smooth.

Existence of integral curves

Theorem 5.27. *Let X be a smooth vector field on M. Then for any p_0 in M there exists an open neighbourhood U of p_0, an $\varepsilon > 0$ and a smooth map*

$$\Gamma :]-\varepsilon, \varepsilon[\times U \to U \subset M$$

such that for any $p \in U$, the (smooth) map

$$\gamma_p :]-\varepsilon, \varepsilon[\xrightarrow{\mathrm{id}\times p}]-\varepsilon, \varepsilon[\times U \xrightarrow{\Gamma} M, \quad \gamma_p(t) = \Gamma(t, p).$$

is an integral curve of X with $\gamma_p(0) = p$. It is moreover unique in the following sense: if $\sigma : I \to M$ is another integral curve of X with $\sigma(0) = p$, then $\gamma_p \equiv \sigma$ on $I \cap]-\varepsilon, \varepsilon[$.

Proof. . Take (U, φ) to be a chart containing p_0. Translate integral curves into system of ODE in $\varphi(U)$, and read Arnold [1] to establish the existence of smooth map $\tilde{\Gamma} :]-\varepsilon, \varepsilon[\times \varphi(U) \to \varphi(U)$ that satisfies the above conditions. $\square$

The result is believable but still leaves one wondering if we can actually extend our solution in t. Even in $\mathbb{R}^n$, it is well-known that this is not always possible.

Example 5.28. Let X be a vector field on $\mathbb{R}$, of the form $X_x = -x^2\partial_x$. Then consider an integral curve problem

$$\gamma'(t) = X_t, \quad \gamma(0) = 1.$$

In other words we are solving the equation $\dot{\gamma} = -\gamma^2$ with an initial condition, and we can try as a solution $\gamma(t) = 1/(t+1)$. This formula makes sense on $]-1, +\infty[$ and one can apply uniqueness arguments to show that this is the only solution. But, we cannot extend it past $-1 \in \mathbb{R}$.

The following result can be generalised, but is a good illustration.

Theorem 5.29. *Let $X \in \mathcal{T}(M)$ be a smooth vector field on M, with compact support. That is, there exists $K \subset M$ compact such that $X \equiv 0$ on $M \setminus K$. Then X is complete: for each $p \in M$ there is an integral curve $\gamma : \mathbb{R} \to M, \gamma(0) = p$ that is defined on the whole of $\mathbb{R}$.*

In particular, on a compact manifold M this is true for any vector field X.

Proof.

1. Fix a compact K outside of which X is zero. Then for any $p \in M \setminus K$, an integral curve of X starting at p simply stays at p, for there $X \equiv 0$. For the same reason an integral curve starting in K cannot exit K.

2. We can conclude using Theorem 5.27 that for each $p \in K$, there is an interval $I_p =]-\varepsilon_p, \varepsilon_p[$ and an integral curve $\gamma_p :]-\varepsilon_p, \varepsilon_p[\to U_p \cap K$ of X with $\gamma_p(0) = p$ and certain uniqueness properties.

3. Choose a finite set of $U_{p_1}, \ldots, U_{p_n}$ such that $K \subset \cup_{p_1,\ldots,p_n} U_p$ (possible since compact) and put $I = \cap_{p_i}]-\varepsilon_{p_i}, \varepsilon_{p_i}[=:]-\varepsilon, \varepsilon[$.

4. Any q in K belongs to some U_{p_i}. Using Theorem 5.27, we conclude again that there is an integral curve $\gamma_q : I \to K$ with $\gamma(0) = K$.

5. Let $p \in K$ and $\gamma : J \to K$ be an integral curve, $\gamma(0) = p$, Let c denote the supremum of J. Denote $q = \gamma(c - \varepsilon/2)$. Then there is an integral curve $\gamma_q : I \to K$.

6. By uniqueness for $|t| < \varepsilon$ we have $\gamma_q(t) = \gamma(t + c - \varepsilon/2)$. It follows that we can extend γ by γ_q on $I \cup]c - \varepsilon/2, c + \varepsilon/2[$.

7. We continue like that in both directions and find unique integral curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ (the uniqueness follows from verifying locally using curves γ_q). $\square$

Example 5.30. Let us revisit Example 5.28. The line $\mathbb{R}$ can be "compactified" to $\mathbb{S}^1$, using the stereographic projection (from $(0, 1)$): there is a chart $\varphi_N : U_N = \mathbb{S}^1 \setminus \{(0, 1)\} \xrightarrow{\sim} \mathbb{R}$ that maps (x, y) to $u = x/(1 - y)$. The "southern pole" stereographic projection φ_S maps (x, y) to $v = x/(1 + y)$ and is defined on $U_S = \mathbb{S}^1 \setminus \{(0, -1)\}$. One computes the transition function on the intersection, discovering $v = 1/u$.

Let us now try to put a vector field X on S^1. On U_N, let us require that $(\varphi_N)_* X|_{U_N} = -u^2 \partial_u$. Then on $U_N \cap U_S$ we must have, considering that $dv = -du/u^2$,

$$(\varphi_S)_* X|_{U_N \cap U_S} = -1/v^2 \cdot (-v^2) \partial_v = \partial_v.$$

Thus we can complete X on the southern chart.

How do we interpret our solution then? We have an integral curve $\varphi_N \circ \gamma(t) = 1/(1 + t)$ that passes through $u = 1$ at $t = 0$. It goes to infinity for $t \to -1$, but it simply means that it tries to escape the chart!

In the southern chart, the integral curve equation looks very simple $(\varphi_S \circ \gamma)'(t) = 1$. Its solutions are given by functions $t \mapsto t + a$.

If we construct $\gamma : \mathbb{R} \to \mathbb{S}^1$ by declaring

$$\gamma(t) = \begin{cases} \varphi_N^{-1}\left(\frac{1}{1+t}\right), & t \in]-1, +\infty[\\ \varphi_S^{-1}(1 + t), & t \in]-\infty, 0[\end{cases}$$

then we have no contradiction on the overlap since there, $1/v = 1/(1+t) = u$. Thus we get an integral curve γ of X with $\gamma(0) = (1, 0)$.

In fact, we simply could have used the second chart to do everything in it! This is the power of perspective change as provided by manifolds.

6 Tensor fields

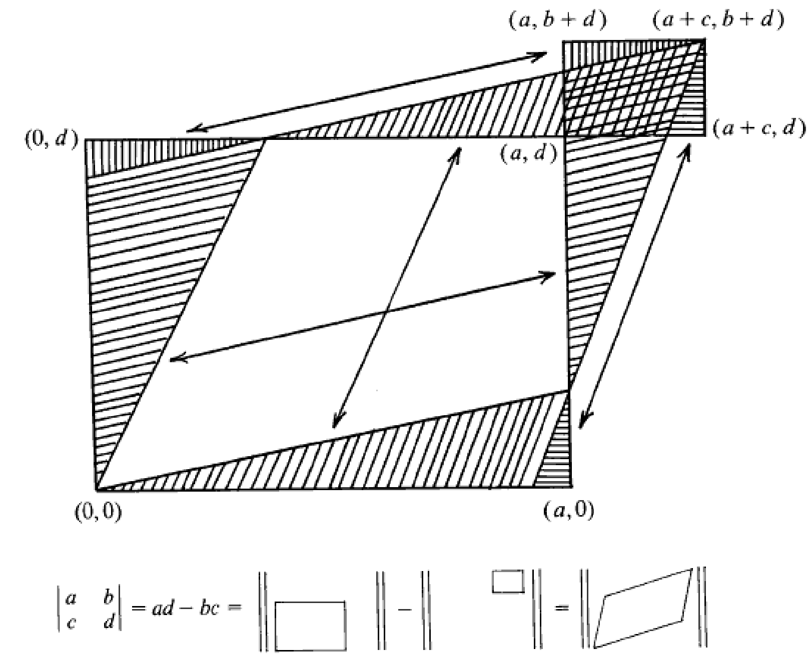
Vector fields are not the only objects that one can study on a manifold. In fact, lots of examples appear as functions of tangent spaces.

At each point $p \in M$, we have $T_p^*M := T_pM^*$, the dual space of the tangent space. It is called the *cotangent space*, its elements are called covectors or 1-forms. We could then attempt to formalise what it means to have a family of 1-forms that varies smoothly with a point.

But we can consider other spaces. For example, the space $\mathcal{B}(T_pM, \mathbb{R})$ of bilinear forms $T_pM \times T_pM \to \mathbb{R}$. There are also linear forms of many arguments that are interesting. For example, we could try to generalise the determinant (antisymmetric multilinear form of rows or columns of a matrix).

Determinants are related to volumes. For example, if we compute $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ it gives, up to a sign, the surface (2-volume) of the parallelogram spanned by (a, b) and (c, d) in $\mathbb{R}^2$.

Proof without words:
A 2 × 2 determinant is the area of a parallelogram



Source: Solomon Golomb, Mathematics Magazine, March 1985.

This statement is true in higher dimensions: the determinant of an $n \times n$-matrix is, up to a sign, the volume of the parallelogram spanned by its rows, or columns.

If we generalise determinants to manifolds, we will have the notion of (infinitesimal) volume as polylinear form on tangent vectors. Indeed, one can develop a theory of integration using these multilinear differential forms!

6.1 Tensors on an $\mathbb{R}$-vector space

Let V be a a vector space over $\mathbb{R}$

Definition 6.1. A *k-tensor* is a map

$$T : V^k = V \times ... \times V \to \mathbb{R}, \quad (v_1, ..., v_k) \mapsto T(v_1, ..., v_k)$$

that is $\mathbb{R}$-linear in each argument: for each $1 \leq i \leq k$ and $v_1, ..., v_{i-1}, v_{i+1}, ..., v_k \in V$, the map

$$w \mapsto T(v_1, ..., v_{i-1}, w, v_{i+1}, ..., v_k) \text{ is } \mathbb{R}\text{-linear.}$$

Just like in the case of bilinear maps, the k-tensors form a vector subspace of the vector space of all functions $\mathcal{F}(V^k, \mathbb{R})$. We denote it $\otimes^k V^*$ (another common notation in the literature: $T^k V^*$).

This notation means in particular that $\otimes^1 V^* = V^*$ the space of linear forms, and technically $\otimes^0 V^* = \mathbb{R}$.

Example 6.2. Let $V = \mathbb{R}^n$. Recall from homework the vector space of matrix tensors $\mathsf{Ten}_n^k(\mathbb{R}) = \{M : \langle n \rangle^k \to \mathbb{R}\}$ with $\langle n \rangle = \{1, ..., n\}$. Let us show that $\otimes^k V^* \cong \mathsf{Ten}_n^k(\mathbb{R})$.

For this, call $e_1, ..., e_n$ the canonical basis of $\mathbb{R}^n$. If we have $v_1, ..., v_k$, then each $v_i = (v_i^1, ..., v_i^n) = \sum_j v_i^j e_j$. Let $T \in \otimes^k V^*$. We can compute

$$T(v_1, ..., v_k) = T\left(\sum_{j_1} v_1^{j_1} e_{j_1}, ..., \sum_{j_k} v_k^{j_k} e_{j_k}\right) = \sum_{j_1, ..., j_k} v_1^{j_1} ... v_k^{j_k} T(e_{j_1}, ..., e_{j_k}).$$

Because of this, for each tensor $T \in \otimes^k V^*$ we define the associated matrix tensor $\mathcal{M}(T)_{j_1 ... j_k} := T(e_{j_1}, ..., e_{j_k})$. Similarly, for each matrix tensor $(j_1, ..., j_k) \mapsto M_{j_1 ... j_k}$, denote

$$\mathcal{T}(M)(v_1, ..., v_k) := \sum_{j_1, ..., j_k} v_1^{j_1} ... v_k^{j_k} M_{j_1 ... j_k};$$

since products of coefficients are multilinear, $\mathcal{T}(M)$ defines a k-tensor. Both $\mathcal{M}$ and $\mathcal{T}$ can be checked to be linear and we have $\mathcal{T}(\mathcal{M}(T)) = T$ and $\mathcal{M}(\mathcal{T}(M)) = M$.

We will continue working with tensors on an abstract vector space without picking a basis, but this example may be useful to keep in mind.

Notation 6.3. Handling multiple arguments is easier if we introduce some index notation. First, we often write sequences like $(v_1, ..., v_k)$, so let us define $v[k] := (v_1, ..., v_k)$ and so $T(v[k]) = T(v_1, ..., v_k)$.

Please note that this notation is not compartible with sums: if we interpreted $v[k] + w[k]$ as $(v_1 + w_1, ..., v_k + w_k)$ then $T(v[k] + w[k]) \neq T(v[k]) + T(w[k])$.

Second, if we have $v[k] = (v_1, ..., v_k)$ and $w[m] = (w_1, ..., w_m)$, write $v[k] * w[m] := (v_1, ..., v_k, w_1, ..., w_m)$. Thus, given any sequence $a[k+m]$ there is unique way to present it as $a[k+m] = v[k] * w[m]$.

Definition 6.4. Let $T \in \otimes^k V^*$ and $P \in \otimes^m V^*$ be two tensors. Their *tensor product* is denoted $T \otimes P \in \otimes^{k+m} V^*$ and is defined as

$$T \otimes P(v[k] * w[m]) := T(v[k]) \cdot P(w[m]), \quad v[k] \in V^k, w[m] \in V^m$$

or in terms of usual indices,

$$T \otimes P(a_1, ..., a_k, a_{k+1}, ..., a_{k+m}) = T(a_1, ..., a_k) P(a_{k+1}, ..., a_{k+m}), \quad a_i \in V.$$

Proposition 6.5. *Let V be a vector space.*

1. *The assignment $(T, P) \mapsto T \otimes P$ defines a bilinear map $\left(\otimes^k V^*\right) \times (\otimes^m V^*) \to \otimes^{k+m} V^*$.*
2. *The tensor product is associative: for all $T \in \otimes^k V^*$, $P \in \otimes^m V^*$ and $Q \in \otimes^l V^*$, one has*
$$(T \otimes P) \otimes Q = T \otimes (P \otimes Q) \text{ in } \otimes^{k+m+l} V^*.$$
Its unit is the 0-tensor corresponding to $1 \in \mathbb{R}$.

One can summarise it by saying that $TV^* := \oplus_{k \geq 0} \otimes^k V^*$ is an $\mathbb{R}$-algebra with respect to $\otimes$.

Proof.

0. The tensor product $T \otimes P$ is a tensor: we develop using the linearity of T or P and then take the product, which is a bilinear map in $\mathbb{R}$.
1. The bilinearity of $\otimes$ amounts to the verifications like this one:
$$((\lambda T_1 + \mu T_2) \otimes P)(v[k] * w[m]) = (\lambda T_1 + \mu T_2)(v[k]) \cdot P(w[m])$$
$$= \lambda T_1(v[k]) \cdot P(w[m]) + \mu T_2(v[k]) \cdot P(w[m]) = (\lambda T_1 \otimes P + \mu T_2 \otimes P)(v[k] * w[m]).$$
2. We note that
$$(v[k] * w[m]) * u[l] = v[k] * (w[m] * u[l]) =: v[k] * w[m] * u[l]$$
is the same sequence, and so
$$[(T \otimes P) \otimes Q]\,((v[k] * w[m]) * u[l]) = (T \otimes P(v[k] * w[m])) \cdot Q(u[l])$$
$$= T(v[k]) \cdot P(w[m]) \cdot Q(u[l])$$
$$= T(v[k]) \cdot (P \otimes Q(w[m] * u[l])) = [T \otimes (P \otimes Q)](v[k] * (w[m] * u[l])).$$
It may look messy, but the idea is very simple: in the end, we are doing a multiplication of functions. For this very reason the 0 argument tensor $1 \in \mathbb{R}$ is the unit for $\otimes$. □

While associative, the tensor product is not in general commutative. We already saw that phenomenon with linear forms.

Pullback as generalisation of transposition

Let $F : V \to W$ be a linear map, then for $v[k] = (v_1, ..., v_k)$ we write $F(v[k]) = (F(v_1), ..., F(v_k))$.

Given $T \in \otimes^k W^*$, define $F^*T \equiv F^*(T)$ by setting

$$F^*(T)(v[k]) := T(F(v[k])) = T(F(v_1), ..., F(v_k)).$$

Lemma 6.6. *The assignment $T \mapsto F^*T$ is a linear map $F^* : \otimes^k W^* \to \otimes^k V^*$, defined for each k. Further, for each two tensors T, T' on W, one has*

$$F^*(T \otimes T') = F^*T \otimes F^*T'.$$

In other words, F^* gives a morphism of $\mathbb{R}$-algebras $TW^* \to TV^*$.

Proof. It is clear that F^*T is a k-tensor on V: we use linearity of F and then multilinearity of T to show multilinearity of F^*T. It is also clear that $F^*(\lambda T + \mu P) = \lambda F^*T + \mu F^*P$: it is the good old pullback of functions. Finally, $F(v[k] * v'[m]) = (F(v_1), ..., F(v'_m)) = F(v[k]) * F(v'[m])$, and

$$F^*(T \otimes T')(v[k] * v'[m]) = (T \otimes T')(F(v[k]) * F(v'[m])) = T(F(v[k]))T'(F(v'[m]))$$

$$= F^*T(v[k])F^*T'(v'[m]) = F^*T \otimes F^*T'(v[k] * v'[m]). \square$$

Basis for $\otimes^k V^*$

We now assume that V is finite dimensional, $\dim V = n$. Denote $f^1, ..., f^n$ a basis in V^*. We have already seen that $f^{i_1} \otimes f^{i_2}$, taken for all i_1, i_2, form a basis of $\otimes^2 V^*$. It is natural to try to take higher tensor products to generalise this statement to $\otimes^k V^*$.

Notation 6.7. A multi-index $(i_1, ..., i_k)$ can be viewed as an element of $\langle n \rangle^k$ with $\langle n \rangle = \{1, ..., n\}$. Similarly to our previous notation I will write $i[k] = (i_1, ..., i_k)$. The only problem is that i and k are too close in the same alphabet, that can lead to confusion.

Because of it, let me write using capital calligraphic letters: $\mathcal{I}[k] = (i_1, ..., i_k)$. Using another letter also permits me to write $\mathcal{I}$ instead of $\mathcal{I}[k]$ if the length is clear or not important. This all is to introduce

$$\eta^{\otimes \mathcal{I}} \equiv \eta^{\otimes \mathcal{I}[k]} := \eta^{i_1} \otimes ... \otimes \eta^{i_k}$$

where $\eta^1, ..., \eta^n$ is any family of linear forms on V.

For any family of vectors $v_1, ..., v_n$ we can similarly write $v[\mathcal{I}] = (v_{i_1}, ..., v_{i_k})$. This can be fed to a k-tensor, $T(v[\mathcal{I}]) = T(v_{i_1}, ..., v_{i_k})$.

Lemma 6.8. *A basis for $\otimes^k V^*$ is given by*

$$\{f^{\otimes \mathcal{I}}\}_{\mathcal{I} \in \langle n \rangle^k} = \{f^{i_1} \otimes ... \otimes f^{i_k}\}_{i_1, ..., i_k \in \langle n \rangle}$$

where $f^1, ..., f^n$ is a basis of V^*.

Proof. Denote the pre-dual basis $e_1, ..., e_n$.

1. Let $\sum_{\mathcal{I}} A_{\mathcal{I}} f^{\otimes \mathcal{I}} = 0$, where as you guessed $A_{\mathcal{I}} = A_{i_1,...,i_k}$. Let $\mathcal{J} = (j_1, ..., j_k)$. We then see

$$f^{\otimes \mathcal{I}}(e[\mathcal{J}]) = f^{i_1}(e_{j_1}) \cdot ... \cdot f^{i_k}(e_{j_k}) = \delta^{i_1}_{j_1} \cdot ... \cdot \delta^{i_k}_{j_k} =: \delta^{\mathcal{I}}_{\mathcal{J}}.$$

   Because of this,

$$0 = \left(\sum_{\mathcal{I}} A_{\mathcal{I}} f^{\otimes \mathcal{I}} \right) (e[\mathcal{J}]) = \sum_{\mathcal{I}} A_{\mathcal{I}} (f^{\otimes \mathcal{I}}(e[\mathcal{J}])) = \sum_{\mathcal{I}} A_{\mathcal{I}} \delta^{\mathcal{I}}_{\mathcal{J}} = A_{\mathcal{J}}.$$

   And so varying $\mathcal{J}$, we get that each $\mathcal{A}_{\mathcal{J}} = 0$. Linear independence is proven.

2. Any $T \in \otimes^k V^*$ can be written as

$$T = \sum_{\mathcal{I}} T(e[\mathcal{I}]) f^{\otimes \mathcal{I}};$$

   this can be verified by tedious computations with multiple sums. □

Remark 6.9. The assignment $\mathcal{I} \mapsto T(e[\mathcal{I}])$ is a k-matrix tensor in the sense of homework 3. Indeed, our verifications show that any choice of basis in V gives an isomorphism $\otimes^k V^* \cong \mathsf{Ten}^k_n(\mathbb{R})$ that sends T to $\{T(e[\mathcal{I}])\}_{\mathcal{I}}$. The image of $f^{\otimes \mathcal{I}}$ under this map is the basis $E^{\mathcal{I}}$ described in the homework.

It is equally true that $T \otimes T'(e[\mathcal{I} * \mathcal{J}]) = T(e[\mathcal{I}]) \cdot T'(e[\mathcal{J}])$, so the isomorphisms $\otimes^k V^* \cong \mathsf{Ten}^k_n(\mathbb{R})$ get promoted to the isomorphism of algebras between TV^* and $\mathsf{Ten}^*_n(\mathbb{R}) = \oplus_k \mathsf{Ten}^k_n(\mathbb{R})$, where the latter is equipped with matrix tensor product.

Remark 6.10. For a linear map $F : V \to W$ and $g^1, ..., g^m$ a basis of W^*, Lemma 6.6 implies that for $\mathcal{I} \in \langle m \rangle^k$, one has $F^*(g^{\otimes \mathcal{I}}) = (F^* g)^{\otimes \mathcal{I}}$. In general, the latter ceases to be a basis. Of course, if F is invertible, then $F^* : \otimes^k W^* \cong \otimes^k V^* : (F^{-1})^*$ are mutually inverse maps, thanks again to Lemma 6.6. In that case taking pullbacks of a basis will produce a basis.

6.2 Tensors on a manifold

We have defined vector fields as families $X_p \in T_p M$ that vary smoothly with a point. To formalise this, we used the fact that we can apply tangent vectors to functions. We then discovered that the same information can be packaged as a derivation of $C^\infty(M)$.

Tensor fields are defined similarly. Remember that for a manifold M, we denote $T^*_p M = (T_p M)^*$. To each $p \in M$, we attach $T_p \in \otimes^k T^*_p M$. We need to say what it means to "vary smoothly with a point"

We note that $T_p : T^*_p M \times ... \times T^*_p M \to \mathbb{R}$ can be applied to a k-tuple of smooth vector fields $X_1, ..., X_k$, if we calculate them at p. The condition is thus to say that

$$p \mapsto T_p((X_1)_p, ..., (X_k)_p)$$

is a smooth function. It works! And in fact, a lot of arguments (but not all) are similar to the vector field case.

Definition 6.11. Let M be a smooth manifold. A *smooth (covariant) k-tensor field* on $U \in \mathsf{Op}\, M$ is a family $T = \{T_p \in \otimes^k T_p^* M\}_{p \in U}$ that satisfies the following condition. For each $V \in \mathsf{Op}\, U$ and each k-tuple of smooth vector fields $X_1, ..., X_k \in \mathcal{T}_M(V)$, the map

$$T(X_1, ..., X_k) : V \to \mathbb{R}, \quad p \mapsto T_p((X_1)_p, ..., (X_k)_p)$$

is smooth. We denote $\otimes^k \mathcal{T}_M^*(U)$ the set of smooth k-tensor fields on U. It is naturally a vector space. To avoid ambiguity, we put $\otimes^0 \mathcal{T}_M^*(U) = C_M^\infty(U)$.

As before, we can use the multi-index notation: $X[k] = (X_1, ..., X_k)$ and $X[k]_p$, $T(X[k])$ and $T_p(X[k]_p)$. If $\mathcal{I} \in \langle \dim M \rangle^k$, then $X[\mathcal{I}] = (X_{i_1}, ..., X_{i_k})$, and so on.

Lemma 6.12. *Let $g \in C_M^\infty(U)$ and $T \in \otimes^k \mathcal{T}^*(U)$. Then $g \cdot T$ defined as*

$$(g \cdot T)(X[k]) := g \cdot T(X[k])$$

is a smooth k-tensor field. The assignment $(g, T) \mapsto g \cdot T$ is bilinear and respects the algebra structure of functions: $g \cdot (h \cdot T) = (g \cdot h) \cdot T$. In other words, $\otimes^k \mathcal{T}_M^*(U)$ is a $C_M^\infty(U)$-module.

Proof. Trivial checks using the smoothness of sum and product operations. $\square$

Another nice property is that we have multilinearity in the strong, C^∞-sense.

Lemma 6.13. *Let $T \in \otimes^k \mathcal{T}_M^*(U)$. Then*

$$T(X_1, ..., fX_i + gY_i, ..., X_k) = fT(X_1, ..., X_i, ..., X_k) + gT(X_1, ..., Y_i, ..., X_k)$$

where $X_1, ..., X_i, Y_i, ..., X_k \in \mathcal{T}_M(V)$ and $f, g \in C_M^\infty(V)$ for $V \subset U$. In particular, T can be viewed as a function

$$T : \mathcal{T}_M(U)^k \to C_M^\infty(U)$$

that is $C_M^\infty(U)$-linear in each argument.

Proof. For each point $p \in V$, we have $(fX_i + gY_i)_p = f(p)(X_i)_p + g(p)(Y_i)_p$. Thus

$$T(X_1, ..., fX_i + gY_i, ..., X_k)(p) = T_p\left((X_1)_p, ..., f(p)(X_i)_p + g(p)(Y_i)_p, ..., (X_k)_p\right)$$

$$= f(p) \cdot T_p\left((X_1)_p, ..., (X_i)_p, ..., (X_k)_p\right) + g(p) \cdot T_p\left((X_1)_p, ..., (Y_i)_p, ..., (X_k)_p\right).$$

Thus we simply used the fact that T_p are multilinear forms of tangent vectors at p. $\square$

Proposition 6.14. *For a manifold M,*

1. *The collection $\{\otimes^k \mathcal{T}_M^*(U)\}_{U \in \mathsf{Op}\, M} = \otimes^k \mathcal{T}_M^*$ is a sheaf of functions to $\otimes^k T^* M := \coprod_p \otimes^k T_p^* M$.*
2. *For $T = \{T_p\}_{p \in U}$ to be smooth, it is sufficient to verify that $p \mapsto T_p(X[k]_p)$ is smooth for any $X[k] \in \mathcal{T}_M(U)^k$ (no need to consider subsets of U).*

3. *As a corollary [4, Lemma 12.24], smooth tensor fields $\otimes^k \mathcal{T}^*_M(U)$ are in bijective correspondence with functions $P : \mathcal{T}_M(U)^k \to C^\infty_M(U)$ that are $C^\infty_M(U)$-linear in each argument. For such a P, its k-tensor at p is given by*

$$P_p(V_1, ..., V_k) = P(X_1, ..., X_k)(p)$$

where $V_1, ..., V_k \in T_pM$ and $X_1, ...X_k$ are any vector fields on U such that $(X_i)_p = V_i$.

The proof of this statement is tedious but not new.

1. It is clear that we can restrict tensor fields, same trick as for vector fields.
2. The sheaf property will follow from the sheaf property of functions.
3. The statements 2. and 3. rely on the extension lemma for vector fields: for each $X \in \mathcal{T}(V)$ there exists a vector field on bigger U that coincides with X in some neighbourhood. This is proven using bump functions.

Perhaps I will add such a proof later.

Example 6.15. For $k = 1$, the elements of $\mathcal{T}^*_M(U) \equiv \otimes^1 \mathcal{T}^*_M(U)$ are called cotangent vector fields, or 1-forms. Perhaps the most important example of a 1-form is simply given by functions. For $f \in C^\infty_M(U)$ and $X \in \mathcal{T}_M(U)$, define df as

$$df : X \mapsto df(X) := X(f), \quad p \mapsto df_p(X_p) := X_p(f) \equiv f_* X_p(\mathrm{id}_\mathbb{R}).$$

There is some double dual game going on here: vectors are functions on functions, so functions are functions on vectors! Each $df_p : Y_p \mapsto Y_p(f)$ is a linear form on T_pM.

Example 6.16. Let $\varphi : U \xrightarrow{\sim} \Omega$ be a chart. We denote $\varphi(p) = (\varphi^1(p), ..., \varphi^n(p))$ where $n = \dim M$. Each $\varphi^i = \varphi^*(x^i)$ is in $C^\infty_M(U)$ as pullback of coordinate functions $x \mapsto x^i$. We claim that $d\varphi^1, ..., d\varphi^n$ generates $\mathcal{T}^*(U)$ under linear sums and multiplication by functions.

Recall the generating vector fields $e^\varphi_1, ..., e^\varphi_n$, $e^\varphi_i(f) := \varphi^*(\partial_{x^i}(f \circ \varphi^{-1}))$. Let us check

$$d\varphi^i(e^\varphi_j) = e^\varphi_j(\varphi^i) = \varphi^*(\partial_{x^j}(\varphi^i \circ \varphi^{-1})) = \varphi^*(\partial_{x^j} x^i) = \varphi^* \delta^i_j = \delta^i_j;$$

here we interpret $\delta^i_j \equiv \delta^i_j \cdot 1$ as a function on U.

Given that any vector field uniquely decomposes as $X = \sum_i X^i e^\varphi_i$ with X^i smooth on U, we have $d\varphi^i(X) = X^i$.

Let $\omega \in \mathcal{T}^*(U)$. Then $\omega(X) = \sum_i \omega(X^i e^\varphi_i)$. By C^∞-linearity we have $\omega(X^i e^\varphi_i) = X^i \omega(e^\varphi_i)$. We conclude that

$$\omega(X) = \sum_i \omega(e^\varphi_i) d\varphi^i(X) = \left(\sum_i \omega(e^\varphi_i) d\varphi^i \right)(X).$$

Of course, any expression of the form $\sum_i \alpha_i d\varphi^i$ with $\alpha^i \in C^\infty(U)$ is also a smooth 1-form. Due to relations $d\varphi^i(e^\varphi_j) = \delta^i_j$ it is easy to check that if $\sum_i \alpha_i d\varphi^i = \sum_i \beta_i d\varphi^i$, then all $\alpha_i = \beta_i$.

One particular case is to consider $U = \Omega$ and $\varphi = \mathrm{id}_\Omega$. Then we have 1-forms $dx^1, ..., dx^n$ that satisfy $dx^i(\partial_{x^j}) = \delta^i_j$. Any $\omega \in \mathcal{T}^*(\Omega)$ can be written as $\omega = \sum \omega_i dx^i$ with ω_i smooth functions on Ω.

For $f \in C^\infty(\Omega)$ and $X = \sum X^i \partial_i$, we have

$$df(X) = X(f) = \sum X^i \partial_i f = \sum \frac{\partial f}{\partial x^i} dx^i(X),$$

and we welcome back the differential formula $df = \sum \partial_{x^i} f \cdot dx^i$.

Remark 6.17. In the example above, one could wonder if in fact any 1-form on M is df for some $f \in C^\infty(M)$. This is true on $M = \mathbb{R}$ (a statement called Poincaré's lemma), but is already false on $M = \mathbb{R}^2 \setminus 0$. A nontrivial example is

$$\omega(x, y) = \frac{xdy - ydx}{x^2 + y^2} = \frac{-y}{x^2 + y^2} \cdot dx + \frac{x}{x^2 + y^2} dy$$

but to prove that $\omega \neq df$, we will need some further theory.

Of course, we can define the tensor product, simply taking tensor products pointwise:

Definition 6.18. Let $T \in \otimes^k \mathcal{T}^*(U)$ and $T' \in \otimes^l \mathcal{T}^*(U)$. Their tensor product is defined as

$$T \otimes T'(X[k] * Y[l]) = T(X[k]) \cdot T'(Y[l]), \quad p \mapsto T_p(X[k]_p) \cdot T'_p(Y[l]_p).$$

The smoothness of products of functions guarantees that $T \otimes T' \in \otimes^{k+l} \mathcal{T}^*(U)$.

Lemma 6.19. *The tensor product satisfies bilinearity and associativity properties similar to those of Proposition 6.5. In addition to that, for each $g \in C^\infty(U)$, one has*

$$(gT) \otimes T' = g(T \otimes T') = T \otimes (gT'). \ \square$$

Example 6.20. Let $\varphi : U \xrightarrow{\sim} \Omega$ be a chart. Then for each $\mathcal{I} \in \langle n \rangle^k$ we can take $d\varphi^{\otimes \mathcal{I}} \in \otimes^k \mathcal{T}^*(U)$. It is easy to see that for any $\mathcal{J} \in \langle n \rangle^k$,

$$d\varphi^{\otimes \mathcal{I}}(e^\varphi[\mathcal{J}]) = d\varphi^{i_1} \otimes ... \otimes d\varphi^{i_k}(e^\varphi_{j_1}, ..., e^\varphi_{j_k}) = \delta^{\mathcal{I}}_{\mathcal{J}}.$$

Because of this, by repeating the same arguments as before, any $T \in \otimes^k \mathcal{T}^*(U)$ can be uniquely written as

$$T = \sum_{\mathcal{J}} T(e[\mathcal{J}]) d\varphi^{\otimes \mathcal{J}} = \sum_{j_1, \dots, j_n} T(e^\varphi_{j_1}, ..., e^\varphi_{j_k}) d\varphi^{j_1} \otimes ... \otimes d\varphi^{j_k}$$

so in short, the tensors on a chart are expressions $\sum_{\mathcal{J}} f_{\mathcal{J}} d\varphi^{\otimes \mathcal{J}}$ with $f_{\mathcal{J}} \in C^\infty(U)$.

We can in particular write $U = \Omega$ and $\varphi = \mathsf{id}$. Then, the k-tensor fields on Ω are simply the expressions $\sum_{\mathcal{J}} f_{\mathcal{J}} dx^{\otimes \mathcal{J}}$ with $f_{\mathcal{J}}$ smooth on Ω.

The sheaf property of tensor fields then gives:

Corollary 6.21. *Let $T : p \mapsto T_p \in \otimes^k T^*_p M$ be a family of k-tensors. Then it is smooth iff for each $p \in M$ there exists a chart (U, φ) containing p such that $T|_U = \sum_{\mathcal{J}} f_{\mathcal{J}} d\varphi^{\otimes \mathcal{J}}$ with $f_{\mathcal{J}}$ smooth on U.*

Pullback

Unlike vector fields, tensor fields can be pulled back along smooth maps $F : M \to N$.

Definition 6.22. Let $F : M \to N$ and $T \in \otimes^k \mathcal{T}^*(N)$. Define $F^*T \equiv F^*(T)$, a family of k-tensors on M, by setting for each $p \in M$

$$F^*(T)_p((X_1)_p, ..., (X_k)_p) := T_{F(p)}(F_*(p)(X_1)_p, ..., F_*(p)(X_k)_p).$$

Here, $(X_i)_p \in T_pM$ and $F_*(p) : T_pM \to T_{F(p)}N$ the pushforward map. The family F^*T is called the *pullback of T along F*.

Proposition 6.23. *In the situation above, one has $F^*G^* = (G \circ F)^*$, and*

1. *For each $f \in C^\infty(N)$, one has $F^*df = d(f \circ F)$,*
2. *The pullback F^*T is a smooth tensor field on M for any $T \in \otimes^k \mathcal{T}^*(N)$,*
3. *The pullback operation is linear is compatible with multiplication of tensors by functions:*

$$F^*(fT_1 + gT_2) = F^*(f) \cdot F^*T_1 + F^*(g) \cdot F^*T_2.$$

More generally, for any two tensor fields T_1, T_2 on N, one has $F^*(T_1 \otimes T_2) = F^*T_1 \otimes F^*T_2$.

Proof.

Let us see what says the first statement. By definition,

$$(F^*df)_p(X_p) = df_{F(p)}(F_*X_p) = F_*X_p(f) = X_p(f \circ F) = d(f \circ F)_p(X_p).$$

Note in particular that this implies that the pullback F^*df is smooth on M.

The verification of C^∞-linearity is a standard pointwise computation. Fix $V_1, ..., V_k$ and $W_1, ..., W_{k'} \in T_pM$. Then:

$$F^*(T \otimes T')_p(V[k] * W[k']) = (T \otimes T')_{F(p)}(F_*(V[k]) * F_*(W[k']))$$
$$= T_{F(p)}(F_*(V[k])) \cdot T'_{F(p)}(F_*(W[k'])) = F^*(T)_p(V[k]) \cdot F^*(T')_p(W[k'])$$
$$= (F^*(T) \otimes F^*(T'))_p(V[k] * W[k']).$$

Its point is that the tensor products are are preserved pointwise. The third statement will thus be proven if we show that F^* preserves smoothness.

Note in particular that

$$F^*(df_1 \otimes ... \otimes df_k) = F^*df_1 \otimes ... \otimes F^*df_k = d(f_1 \circ F) \otimes ... \otimes d(f_k \circ F);$$

everything here is smooth, so F^* preserves smoothness of the decomposable k-tensors. This suggests that we might want to do it in charts.

First, consider the situation

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\bar{F}"'] & M \arrow[d, "F"] \\ V \arrow[r] & N \end{tikzcd}$$

with U, V open and $\bar{F} = F|_U$. For $T \in \otimes^k \mathcal{T}^*(N)$, we claim that $(F^*T)|_U = \bar{F}^*(T|_V)$. This is true at each point p of U, since the diagram on the level of tangent spaces commutes:

$$\begin{tikzcd} T_pU \arrow[r, "\sim"] \arrow[d, "\bar{F}_*"'] & T_pM \arrow[d, "F_*"] \\ T_{F(p)}V \arrow[r, "\sim"'] & T_{F(p)}N. \end{tikzcd}$$

In detail, for $X_1, ..., X_k \in T_pU = T_pM$, using $T_pV = T_pN$.

$$(F^*T)_p(X[k]) = T_{F(p)}(F_*X[k]) = T_{F(p)}(\bar{F}_*X[k]) = (\bar{F}^*T|_V)_p(X[k]).$$

Finally, we for each $p \in M$ there exist charts (U, φ) containing p and (V, ψ) containing $F(p)$, and a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\bar{F}"'] & M \arrow[d, "F"] \\ V \arrow[r] & N \end{tikzcd}$$

Any smooth k-tensor field P on V can be written as $P = \sum_{\mathcal{I}} f_{\mathcal{I}} d\psi^{\otimes \mathcal{I}}$. We verify that

$$\bar{F}^*(P) = \bar{F}^*\left(\sum_{\mathcal{I}} f_{\mathcal{I}} d\psi^{\otimes \mathcal{I}}\right) = \sum_{\mathcal{I}} F^*(f_{\mathcal{I}}) \cdot \bar{F}^* d\psi^{\otimes \mathcal{I}}$$

The latter expression is a smooth k-tensor field on U. For a general T on N, we can say that there exists a cover $\cup U_i = M$ such that $F^*T|_{U_i}$ is smooth for each i. Sheaf property allows to conclude. $\square$

Example 6.24. The proof above really explains how to compute the pullbacks. In particular, let us consider a map $F : \Omega \to \Theta$. Denote $(x^1, ..., x^n)$ the coordinates in Ω and $(y^1, ..., y^m)$ the coordinates in Θ. If we write $F(x) = (F^1(x), ..., F^m(x))$, then $y^i \circ F = F^i$, and so $F^*(dy^i) = d(y^i \circ F) = dF^i$.

We already computed that at $p = (x^1, ..., x^n)$, one has $dF^i_p = \sum \frac{\partial F^i}{\partial x^j}(p) dx^j_p$, or simply $dF^i = \sum_j \partial_{x^j} F^i \cdot dx^j$. For each $\mathcal{I} \in \langle m \rangle^k$, we thus get

$$\begin{aligned} F^*(dy^{\otimes \mathcal{I}}) &= dF^{\otimes \mathcal{I}} = dF^{i_1} \otimes ... \otimes dF^{i_k} \\ &= \sum_{1 \le j_1, ..., j_k \le n} \frac{\partial F^{i_1}}{\partial x^{j_i}} \cdot ... \cdot \frac{\partial F^{i_k}}{\partial x^{j_k}} dx^{j_1} \otimes ... \otimes dx^{j_k} \\ &= \sum_{\mathcal{J} \in \langle n \rangle^k} J(F)^{\mathcal{I}}_{\mathcal{J}} dx^{\otimes \mathcal{J}}, \quad J(F)^{\mathcal{I}}_{\mathcal{J}} = \frac{\partial F^{i_1}}{\partial x^{j_i}} \cdot ... \cdot \frac{\partial F^{i_k}}{\partial x^{j_k}} \end{aligned}$$

As a consequence, given any $T = \sum T_{\mathcal{I}} dy^{\otimes \mathcal{I}}$ with $T_{\mathcal{I}}$ smooth functions on Θ (matrix k-tensors attached to each point of Θ), we have

$$(F^*T)_p = \sum_{\mathcal{I} \in \langle m \rangle^k, \mathcal{J} \in \langle n \rangle^k} T_{\mathcal{I}}(F(p)) \cdot J(F)^{\mathcal{I}}_{\mathcal{J}}(p) \cdot dx_p^{\otimes \mathcal{J}}.$$

Arguably, the theory of k-tensors looks like a generalisation of the calculus of differentials to "many entries", and the formulas that we achieve are very natural, simply the "added up" chain rules. There are approaches to differential geometry that define 1-forms and tensor fields *before* vector fields as certain formal algebraic expressions, but there are some hidden difficulties with that.

Let us conclude this section with defining an example that is key to a whole subdomain of differential geometry.

Definition 6.25. A *Riemannian metric* on a manifold M is a smooth 2-tensor field $g \in \otimes^2 \mathcal{T}^*(M)$ such that at each point $p \in M$, the map

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

defines an inner product on T_pM.

Example 6.26. On $\mathbb{R}^n$, we can consider $g = dx^1 \otimes dx^1 + ... + dx^n \otimes dx^n$. Given two tangent vectors $X = \sum X^i \partial_i|_p$, $Y = \sum Y^i \partial_i|_p$ at p, we see, using $dx^i_p(\partial_j|_p) = \delta^i_j$,

$$g_p(X, Y) = \sum_{1 \leq i \leq n} dx^i_p(X) dx^i_p(Y) = \sum_{1 \leq i \leq n} X^i Y^i$$

and this is indeed an inner product.

Example 6.27. It should be familiar to you from MAA201 or MAA206 that if $i : V \to W$ is an injection of $\mathbb{R}$-vector spaces and h is an inner product on W, then $(v, v') \mapsto h(i(v), i(v'))$ is an inner product on V.

Thus if we have an immersion $F : M \to N$ and g is a Riemannian metric on N, then $F^*(g)$ will be a Riemannian metric on M. Indeed, for $X, Y \in T_pM$,

$$(F^*g)_p(X, Y) = g_{F(p)}(F_*X, F_*Y)$$

and by definition F_* is an injection.

This can in particular be applied to submanifolds of Euclidean spaces. For $i : \mathbb{S}^n \subset \mathbb{R}^{n+1}$, we can pull back $g = dx^0 \otimes dx^0 + ... + dx^n \otimes dx^n$ to obtain $g_r := i^*g$, the Riemannian metric on $\mathbb{S}^n$, called the round metric.

If you want to know what it looks like in a coordinate chart $\varphi : U \cong \Omega$, the conventional way is to consider

$$(\varphi^{-1})^*(g_r|_U) = (\varphi^{-1})^*(i^*g)|_U = (i \circ \varphi^{-1})^*g.$$

This we can compute since $i \circ \varphi^{-1} : \Omega \to \mathbb{R}^{n+1}$ is a smooth map between opens in Euclidean spaces.

Riemannian metrics are extremely useful for a variety of applications, and in fact some Riemannian metric always exist on a (paracompact) manifold. There is a generalisation to Lorentzian metrics that is used in general relativity.

Let I be an open interval in $\mathbb{R}$ and $\gamma : I \to M$ a smooth map. At each point $t \in I$, we have $\frac{d}{dx}|_t$, and as we explained one can consider the tangent vector to the curve γ:

$$\gamma'(t) := \gamma_*(t)\left(\left.\frac{d}{dx}\right|_t\right) \in T_{\gamma(t)}M.$$

If g is a Riemannian metric on M, then we can define

$$\left\|\gamma'(t)\right\|_g := \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} \geq 0$$

and for $a \leq b$ two points of I, we can define the length of curve γ as

$$L_g(\gamma) = \int_a^b \left\|\gamma'(t)\right\|_g dt$$

and it is possible, using pullbacks of g, to show that the function $t \mapsto \|\gamma'(t)\|_g$ is smooth. This may seem abstract but with some practice, it allows to compute curve lengths on the familiar examples!

Example 6.28. Let us simply decipher what this all means in the case of $\mathbb{R}^n$ with $g = \sum dx^i \otimes dx^i$. A curve $\gamma : I \to \mathbb{R}^n$ is a collection of smooth maps $t \mapsto (\gamma^1(t), ...\gamma^n(t))$. In this case

$$\gamma'(t)(f) = \frac{d(f \circ \gamma)}{dt}(t) = \sum_i \frac{d\gamma^i}{dt}(t)\frac{\partial f}{\partial x^i}(\gamma(t)).$$

In other words $\gamma'(t) = \sum_i \dot{\gamma}^i(t)\partial_i|_{\gamma(t)}$. Feeding this to g gives

$$\left\|\gamma'(t)\right\|_g = \sqrt{(\dot{\gamma}^1(t))^2 + ... + (\dot{\gamma}^n(t))^2},$$

something that is familiar to physicists as the velocity vector. The integrals

$$\int_a^b \left\|\gamma'(t)\right\|_g dt = \int_a^b \sqrt{(\dot{\gamma}^1(t))^2 + ... + (\dot{\gamma}^n(t))^2} dt$$

have a natural interpretation of the length of the "physical" trajectory in the space $\mathbb{R}^n$.

7 Differential forms

7.1 Alternating forms on a vector space

We use the notation $\Sigma_k = \mathsf{Aut}(\langle k \rangle)$ to denote the group of permutations of k elements:

$$\sigma : 1, ..., k \mapsto \sigma(1), ..., \sigma(k).$$

We shall make use of compositions of permutations: for $\mu, \sigma \in \Sigma_k$, the composed permutation $\mu \circ \sigma$ acts as $i \mapsto \mu(\sigma(i))$.

Definition 7.1. Let V be a $\mathbb{R}$-vector space. A k-tensor $A \in \otimes^k V^*$ is *alternating*, or *skew-symmetric*, or a *linear k-form*, if for each $v_1, ..., v_k \in V$ and each $\sigma \in \Sigma_k$, we have

$$A(v_1, ..., v_k) = (-1)^\sigma A(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

where $(-1)^\sigma = +1$ if σ can be expressed as an even number of transpositions, and $(-1)^\sigma = -1$ otherwise.

We can again introduce some multi-index notation. For $v[k] = (v_1, ...v_k)$, we interpret it as a function $\langle k\rangle \to V$ and write $\sigma^* v[k] = (v_{\sigma(1)}, ..., v_{\sigma(k)})$. Thus the above equality becomes $A(v[k]) = (-1)^\sigma A(\sigma^* v[k])$.

We denote $\Lambda^k V^* \subset \otimes^k V^*$ the set of all k-linear forms. It is easy to see that it is a vector subspace. To understand it better, we would like to consider a projection from $\otimes^k V^*$ to $\Lambda^k V^*$.

For $\sigma \in \Sigma_k$ and $T \in \otimes^k V^*$, denote $\sigma_* T$ the tensor

$$\sigma_* T(v[k]) = (-1)^\sigma T(\sigma^* v[k]).$$

Lemma 7.2. *The assignment $T \mapsto \sigma_* T$ defines a linear map $\sigma_* : \otimes^k V^* \to \otimes^k V^*$. Moreover, for $\sigma, \mu \in \Sigma_k$, we have $\mu_*(\sigma_* T) = (\mu \circ \sigma)_* T$. In other words, $\sigma \mapsto \sigma_*$ provides a representation of Σ_k on the k-tensors.*

Proof. The linearity of σ_* is easy to verify. We now try to understand how it works with respect to compositions. By definition,

$$\mu_*(\sigma_* T)(v[k]) = (-1)^\mu \sigma_* T(\mu^* v[k]) = (-1)^\mu \sigma_* T(v_{\mu(1)}, ..., v_{\mu(k)}).$$

Denote $w_i = v_{\mu(i)}$. Then $w_{\sigma(i)} = v_{\mu(\sigma(i))}$ and so

$$(-1)^\mu \sigma_* T(w_1, ..., w_k) = (-1)^\mu (-1)^\sigma T(v_{\mu(\sigma(1))}, ..., v_{\mu(\sigma(k))}).$$

We can conclude that

$$\mu_* \sigma_* T(v[k]) = (-1)^{\mu\circ\sigma} T((\mu \circ \sigma)^* v[k])$$

since the signs of permutations are also compatible with composition (so-called sign representation of Σ_k). □

Definition 7.3. For $T \in \otimes^k V^*$, denote

$$\mathsf{Alt}(T) := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \sigma_* T.$$

In other words,

$$\mathsf{Alt}(T)(v_1, ..., v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^\sigma T(v_{\sigma(1)}, ..., v_{\sigma(k)}).$$

Lemma 7.4. *The operation $T \mapsto \mathsf{Alt}(T)$ is a linear map $\mathsf{Alt} : \otimes^k V^* \to \Lambda^k V^*$ that satisfies $\mathsf{Alt}(A) = A$ for each $A \in \Lambda^k V^*$. In particular, $\mathsf{Alt} \circ \mathsf{Alt} = \mathsf{Alt}$.*

Proof.

1. Note that $A \in \Lambda^k V^*$ iff for each $\sigma \in \Sigma_k$, one has $\sigma_* A = A$ (A is an invariant of the representation). For this reason
$$\mathsf{Alt}(A) = \frac{1}{k!}\sum_{\sigma} \sigma_* A = \frac{1}{k!}\sum_{\sigma} A = A.$$

2. The operation $T \mapsto \mathsf{Alt}(T)$ is linear as a linear combination of linear maps. To see that it lands in $\Lambda^k V^*$, note that
$$\mu_* \mathsf{Alt}(T) = \mu_*\left(\frac{1}{k!}\sum_{\sigma}\sigma_* T\right) = \frac{1}{k!}\sum_{\sigma}(\mu\circ\sigma)_* T.$$
For each $\tau \in \Sigma_k$, there exists unique σ such that $\tau = \mu \circ \sigma$: just take $\sigma = \mu^{-1}\circ\tau$. Thus in the sum above, each permutation $\mu\circ\sigma = \tau \in \Sigma_k$ will appear exactly once. This means that we can write
$$\frac{1}{k!}\sum_{\sigma}(\mu\circ\sigma)_* T = \frac{1}{k!}\sum_{\tau}\tau_* T = \mathsf{Alt}(T)$$
This concludes the proof that $\mu_* \mathsf{Alt} = \mathsf{Alt}$. □

The operation $T \mapsto \mathsf{Alt}(T)$ is defined for each $k \geq 0$ and is identity for $k \leq 1$. To understand it better, we shall study the interaction of Alt with tensor products.

Proposition 7.5. *Let $T \in \otimes^k V^*$, $P \in \otimes^m V^*$ and $Q \in \otimes^l V^*$ be three tensors, then*

1. $\mathsf{Alt}(T \otimes P) = (-1)^{km}\mathsf{Alt}(P \otimes T)$,
2. $\mathsf{Alt}(\mathsf{Alt}(T\otimes P)\otimes Q) = \mathsf{Alt}(T\otimes P\otimes Q) = \mathsf{Alt}(T\otimes \mathsf{Alt}(P\otimes Q))$.

Proof.

1. Denote $s \in \Sigma_{k+m}$ the following permutation:
$$s(i) = m + i, 1 \leq i \leq k, \quad s(j) = j - k, k+1 \leq j \leq k+m;$$
in other words, s pushes first k elements to become last k elements, without changing order. If we imagine it via transpositions then one needs to pass each $k, k-1, ..., 1$ through m elements. In other words, $(-1)^s = (-1)^{km}$.

2. Because of how we defined s, it is easy to see that
$$s_*(T\otimes P) = (-1)^s P \otimes T.$$

3. Using this, we compute
$$(-1)^{km}\mathsf{Alt}(P\otimes T) = \mathsf{Alt}(s_*(T\otimes P)) = \frac{1}{(k+m)!}\sum_{\sigma}(\sigma\circ s)_*(T\otimes P).$$
We can then reason the same way as in Lemma 7.4 to conclude that the latter sum is equal to $\mathsf{Alt}(T\otimes P)$.

4. Denote $R := T \otimes P \in \otimes^r V^*$ where $r = k + m$. We now prove that $\mathsf{Alt}(\mathsf{Alt}(R) \otimes Q) = \mathsf{Alt}(R \otimes Q)$. Note that
$$\mathsf{Alt}(R) \otimes Q = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} (\sigma_* R) \otimes Q = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \tilde{\sigma}_*(R \otimes Q)$$
where $\tilde{\sigma}$ is a $r + l$-permutation that acts as σ on the first r elements.

5. The set of all permutations of Σ_{r+l} that are identity on the last l elements is a subgroup $H \subset \Sigma_{r+l}$ isomorphic to Σ_r via the assignment $\sigma \mapsto \tilde{\sigma}$.

6. With this in mind, we have
$$\mathsf{Alt}(\mathsf{Alt}(R) \otimes Q) = \frac{1}{(r+l)!} \frac{1}{r!} \sum_{\mu \in \Sigma_{r+l}} \sum_{\tilde{\sigma} \in H} (\mu \circ \tilde{\sigma})_*(R \otimes Q)$$
$$= \frac{1}{(r+l)!} \frac{1}{r!} \sum_{\tilde{\sigma} \in H} \sum_{\tau \in \Sigma_{r+l}} \tau_*(R \otimes Q) = \frac{1}{(r+l)!} \frac{r!}{r!} \sum_{\tau \in \Sigma_{r+l}} \tau_*(R \otimes Q).$$
Here we interchanged the sum order and understood, again, that $\sum_{\mu \in \Sigma_{r+l}} (\mu \circ \tilde{\sigma})_*(R \otimes Q) = \sum_{\tau \in \Sigma_{r+l}} \tau_*(R \otimes Q)$ just like in Lemma 7.4.

7. This concludes the proof of the second point of the proposition, as we can reason the similar way for the other equation. □

As you can see, all the index computations disappeared once the we formalised the action of symmetric groups on spaces of tensors.

Definition 7.6. For $A \in \Lambda^k V^*$ and $B \in \Lambda^m V^*$, we define their *wedge product* $A \wedge B$ as
$$A \wedge B := \frac{(k+m)!}{k!m!} \mathsf{Alt}(A \otimes B).$$
In particular, let us take $f, f' \in \Lambda^1 V^* = V^*$. Then
$$f \wedge f' = \frac{2!}{1!1!} \frac{1}{2!} (f \otimes f' - f' \otimes f) = f \otimes f' - f' \otimes f.$$

Lemma 7.7. *The wedge product satisfies:*

1. *It is a bilinear map* $\wedge : \Lambda^k V^* \times \Lambda^m V^* \to \Lambda^{k+m} V^*$,
2. *It is graded-commutative:* $A \wedge B = (-1)^{km} B \wedge A$ *for* $A \in \Lambda^k V^*$ *and* $B \in \Lambda^m V^*$,
3. *It is associative:* $(A \wedge B) \wedge C = A \wedge (B \wedge C)$.

Proof. It is probably worth showing how the coefficients work for associativity:
$$(A \wedge B) \wedge C = \frac{(k+m)!}{k!m!} \mathsf{Alt}(A \otimes B) \wedge C = \frac{(k+m+l)!}{(k+m)!l!} \frac{(k+m)!}{k!m!} \mathsf{Alt}(\mathsf{Alt}(A \otimes B) \otimes C)$$
$$= \frac{(k+m+l)!}{k!m!l!} \mathsf{Alt}(A \otimes B \otimes C)$$
and a similar computation for the other triple product. □

Remark 7.8. Repeating the previous computation inductively, one can show that given $A_1, ..., A_m$, $A_i \in \Lambda^{k_i} V^*$, one has

$$A_1 \wedge ... \wedge A_m = \frac{(k_1 + ... + k_m)!}{k_1!...k_m!} \operatorname{Alt}(A_1 \otimes ... \otimes A_m).$$

In particular, given m 1-forms $f^1, ..., f^m$, we have $f^1 \wedge ... \wedge f^m = m! \operatorname{Alt}(f^1 \otimes ... \otimes f^m)$, or

$$f^1 \wedge ... \wedge f^m = \sum_{\sigma \in \Sigma_m} (-1)^\sigma f^{\sigma(1)} \otimes ... \otimes f^{\sigma(m)}.$$

Corollary 7.9. *Let $\omega \in \Lambda^{2k+1} V^*$. Then $\omega \wedge \omega = 0$.*

Proof. $\omega \wedge \omega = (-1)^{(2k+1)^2} \omega \wedge \omega = -\omega \wedge \omega$. □

A word about linear maps.

Lemma 7.10. *Let $F : V \to W$ be a linear map, then*

1. *The assignment $A \mapsto F^* A$ defines a linear map $\Lambda^k W^* \to \Lambda^k V^*$. In other words, the pullback of k-tensors preserves skew-symmetry.*
2. *One has $F^*(A \wedge B) = F^* A \wedge F^* B$.*

Proof. Since $F^* A(v_1, ..., v_k) = A(F(v_1), ..., F(v_k))$, it is easy to see that this expression is skew-symmetric. Moreover, we see that

$$\begin{aligned} &\operatorname{Alt}(F^* A \otimes F^* B)(v_1, ..., v_{k+m}) \\ &= \frac{1}{(k+m)!} \sum_\sigma (-1)^\sigma A(F(v_{\sigma(1)}), ..., F(v_{\sigma(k)})) B(F(v_{\sigma(k+1)}), ..., F(v_{\sigma(k+m)})) \\ &= \operatorname{Alt}(A \otimes B)(F(v_1), ..., F(v_k), F(v_{k+1}), ..., F(v_{k+m})) \\ &= F^* \operatorname{Alt}(A \otimes B)(v_1, ..., v_{k+m}). \end{aligned}$$

We then conclude using the definition of $\wedge$. □

Basis for $\Lambda^k V^*$

Notation 7.11. Given $f^1, ..., f^n$, a family of linear forms on V, we can introduce, for $\mathcal{I} = (i_1, ..., i_k) \in \langle n \rangle^k$, the notation

$$f^{\wedge \mathcal{I}} := f^{i_1} \wedge ... \wedge f^{i_k}.$$

In light of the previous corollary and Lemma 7.7, we see that if $\mathcal{I}$ has a repeating index, then $f^{\wedge \mathcal{I}} = 0$. In effect, if for example $i_m = i_{m'}$, then by Corollary 7.9

$$f^{i_1} \wedge ... \wedge f^{i_k} = \pm f^{i_m} \wedge f^{i_{m'}} \wedge ... = 0.$$

We are thus inclined to consider only the *ordered* $\mathcal{I}$, those multi-indexes for which $1 \le i_1 < i_2 < ... < i_k \le n$. Even in this case we might still get $f^{\wedge \mathcal{I}} = 0$ if the f^i are linearly dependent.

Proposition 7.12. *Let $f^1, ..., f^n$ be a basis of V^*. Then*

$$f^{\wedge\mathcal{I}},\ \mathcal{I} = (i_1, ..., i_k),\ 1 \le i_1 < ... < i_k \le n$$

form a basis of $\Lambda^k V^*$. In particular, its dimension is $\binom{n}{k}$, meaning that for $k > n$, $\Lambda^k V^* = 0$.

Proof.

1. Choose $e_1, ..., e_n$ the pre-dual basis to $f^1, ..., f^n$. For each ordered $\mathcal{I}$ and each ordered $\mathcal{J} = (j_1, ..., j_k)$, we have

$$f^{\wedge\mathcal{I}}(e[\mathcal{J}]) = \sum_\sigma (-1)^\sigma f^{\otimes\sigma^*\mathcal{I}}(e[\mathcal{J}]) = f^{i_1}(e_{j_1}) \otimes ... \otimes f^{i_k}(e_{j_k}) + 0 = \delta^{\mathcal{I}}_{\mathcal{J}}.$$

   Here we use that for σ different from identity, $f^{i_{\sigma(1)}}(e_{j_1}) \otimes ... \otimes f^{i_{\sigma(k)}}(e_{j_k})$ is always zero. Otherwise we would have $(j_1, ..., j_k) = (i_{\sigma(1)}, ..., i_{\sigma(k)})$, but this is impossible since $\sigma^*\mathcal{I} = (i_{\sigma(1)}, ..., i_{\sigma(k)})$ is never ordered.

2. We use this to show linear independence: if

$$\sum_{\mathcal{I},<} C_{\mathcal{I}} f^{\wedge\mathcal{I}} := \sum_{\mathcal{I},1\le i_1<...<i_k\le n} C_{\mathcal{I}} f^{\wedge\mathcal{I}} = 0$$

   then we evaluate it on all $e[\mathcal{J}]$ for all ordered $\mathcal{J} \in \langle n\rangle^k$ to get that $C_{\mathcal{J}} = 0$.

3. Any $A \in \Lambda^k V^*$ is also a k-tensor, so it can be written as $\sum_{\mathcal{I}} A_{\mathcal{I}} f^{\otimes\mathcal{I}}$, taken over all multi-indices. Now,

$$A = \mathsf{Alt}(A) = \sum_{\mathcal{I}} A_{\mathcal{I}} \mathsf{Alt}(f^{\otimes\mathcal{I}}) = \frac{1}{k!}\sum_{\mathcal{I}} A_{\mathcal{I}} f^{\wedge\mathcal{I}}.$$

   The proof is over, since either $f^{\wedge\mathcal{I}} = 0$ if $\mathcal{I}$ contains a repeating index, or $f^{\wedge\mathcal{I}} = \pm f^{\wedge\mathcal{J}}$ where $\mathcal{J}$ is ordered. □

Remark 7.13. We see that each k-form A can be written in two ways:

$$A = \sum_{\mathcal{I},<} A_{\mathcal{I}} f^{\wedge\mathcal{I}} = \frac{1}{k!}\sum_{\mathcal{I}} B_{\mathcal{I}} f^{\wedge\mathcal{I}}.$$

The former presentation is unique, since it is a decomposition with respect to a basis. The coefficients $A_{\mathcal{I}}$ can be computed as $A(e[\mathcal{I}])$. The latter presentation is not unique. For example,

$$f^1 \wedge f^2 = \frac{1}{2}f^1 \wedge f^2 - \frac{1}{2}f^2 \wedge f^1 = \frac{1}{2}\left(\frac{1}{2}f^1 \wedge f^2 - \frac{3}{2}f^2 \wedge f^1\right).$$

We can fix things somewhat by putting $B_{\mathcal{I}} = A_{\mathcal{I}}$ when $\mathcal{I}$ is ordered, by putting $B_{j_1...j_k} = 0$ if some index is repeating, and setting in all other cases $B_{j_1...j_k} = (-1)^\sigma A_{i_1...i_k}$ where $(i_1, ..., i_k)$ is the ordered multi-index uniquely obtained from $(j_1, ..., j_k)$ by some permutation σ. This is, however, a *choice*: in particular, one can take any values for coefficients $B_{\mathcal{I}}$ with repeating indices.

One nice application of bases is the following lemma:

Lemma 7.14. *Let V be a vector space of dimension n. For any linear map $F : V \to V$ and any $\omega \in \Lambda^n V^*$, one has $F^*\omega = \det F \cdot \omega$ where $\det F$ is defined, as usual, as $\det$ of the matrix of F in some basis.*

Proof. Take a basis $e_1, ..., e_n$, denote $M = (M^i_j)$ the matrix of F with respect to that basis: $F(e_i) = \sum_j e_j M^j_i$. Denote $f^1, ..., f^n$ the dual basis. From MAA201 (Ch.1, Prop 4.3) we know that $F^*(f^i) = \sum_j M^i_j f^j$. Thus,

$$F^*(f^1 \wedge ... \wedge f^n) = F^*(f^1) \wedge ... \wedge F^*(f^n) = \sum_{j_1,...j_n} M^1_{j_1} \cdot M^n_{j_n} f^{j_1} \wedge ... \wedge f^{j_n}.$$

We observe that either $f^{j_1} \wedge ... \wedge f^{j_n} = 0$ (some index is repeating), or $f^{j_1} \wedge ... \wedge f^{j_n} = (-1)^\sigma f^1 \wedge ... \wedge f^n$, where σ satisfies $\sigma(i) = j_i$. For this reason we can write

$$F^*(f^1 \wedge ... \wedge f^n) = \sum_\sigma (-1)^\sigma M^1_{\sigma(1)} \cdot M^n_{\sigma(n)} f^1 \wedge ... \wedge f^n = \det M \cdot f^1 \wedge ... \wedge f^n.$$

The proof is concluded by observing that any $\omega \in \Lambda^n V^*$ is uniquely expressed as $\omega = c \cdot f^1 \wedge ... \wedge f^n$. □

Note that any $\omega \neq 0$ serves as a basis of $\Lambda^n V^*$ since the dimension of the latter is $\binom{n}{n} = 1$. One could reverse the discussion and define $\det F$ using the identity of the lemma.

7.2 Differential forms on manifolds

Definition 7.15. Let M be a smooth manifold. A *differential k-form* on $U \in \mathsf{Op}\, M$ is a family $\omega = \{\omega_p \in \Lambda^k T^*_p M\}_{p \in U}$ that is a smooth k-tensor. That is, for each $V \in \mathsf{Op}\, U$ and each k-tuple of smooth vector fields $X_1, ..., X_k \in \mathcal{T}_M(V)$, the map

$$\omega(X_1, ..., X_k) : V \to \mathbb{R}, \quad p \mapsto \omega_p((X_1)_p, ..., (X_k)_p)$$

is smooth. We denote $\Lambda^k_M(U)$ the set of differential k-forms on U. To avoid ambiguity, we put $\Lambda^0_M(U) := C^\infty_M(U)$.

As before, we can use the multi-index notation: $X[k] = (X_1, ..., X_k)$ and $X[k]_p$, $\omega(X[k])$ and $\omega_p(X[k]_p)$. If $\mathcal{I} \in \langle \dim M \rangle^k$, then $X[\mathcal{I}] = (X_{i_1}, ..., X_{i_k})$, and so on.

Lemma 7.16. *The differential k-forms $\Lambda^k_M(U)$ form a vector subspace of $\otimes^k \mathcal{T}^*(U)$. Moreover, for $g \in C^\infty_M(U)$ and $\omega \in \Lambda^k_M(U)$, the product $g \cdot \omega$ defined previously as*

$$(g \cdot \omega)(X[k]) := g \cdot \omega(X[k])$$

is a differential k-form. In other words, $\Lambda^k_M(U)$ is a $C^\infty_M(U)$-submodule of $\otimes^k \mathcal{T}^*_M(U)$.

Proof. Mulitplying by a function does not break alternation. □

Being k-tensors, the differential forms enjoy the multilinearity in the C^∞-sense: for $\omega \in \Lambda^k_M(U)$,

$$\omega(X_1, ..., fX_i + gY_i, ..., X_k) = f\omega(X_1, ..., X_i, ..., X_k) + g\omega(X_1, ..., Y_i, ..., X_k)$$

where $X_1, ..., X_i, Y_i, ..., X_k \in \mathcal{T}_M(V)$ and $f, g \in C^\infty_M(V)$ for $V \subset U$. In particular, ω, viewed as a function from $\mathcal{T}_M(U)^k$ to $C^\infty_M(U)$, is $C^\infty_M(U)$-linear in each argument.

Proposition 7.17. *For a (smooth) manifold M,*

1. *The collection $\{\Lambda^k_M(U)\}_{U \in \mathsf{Op}\, M} = \Lambda^k_M$ is a sheaf of functions to $\Lambda^k T^*M := \coprod_p \Lambda^k T^*_p M \subset \coprod_p \otimes^k T^*_p M$.*
2. *For $\omega = \{\omega_p\}_{p \in U}$ to be smooth, it is sufficient to verify that $p \mapsto \omega_p(X[k]_p)$ is smooth for any $X[k] \in \mathcal{T}_M(U)^k$ (no need to consider subsets of U).*
3. *As a corollary, differential k-forms $\Lambda^k_M(U)$ are in bijective correspondence with functions $A : \mathcal{T}_M(U)^k \to C^\infty_M(U)$ that are $C^\infty_M(U)$-linear in each argument and are alternating. For such an A, its k-form at p is given by*

   $$A_p(V_1, ..., V_k) = A(X_1, ..., X_k)(p)$$

   where $V_1, ..., V_k \in T_pM$ and $X_1, ...X_k$ are any vector fields on U such that $(X_i)_p = V_i$ (they can be shown to exist).

Proof. The proof of this statement is not very hard if we accept the similar Proposition 6.14 for the tensors.

1. A restriction of an alternating tensor $\omega \in \Lambda^k_M(U)$ to $V \subset U$ is alternating at each point of V. Similarly, a tensor T such that T_p is alternating at each point p of U_i, where $\cup U_i = U$, is alternating everywhere.
2. Being smooth as a differential k-form is the same as being smooth as a k-tensor.
3. The function A will correspond to some k-tensor. One then verifies from the alternating property of A that each A_p is alternating. $\square$

Example 7.18. For $k = 1$, we have $\Lambda^1_M(U) = \mathcal{T}^*_M(U) \equiv \otimes^1 \mathcal{T}^*_M(U)$. Thus for we have the example of df defined for $f \in C^\infty_M(U) = \Lambda^0_M(U)$ as

$$df : X \mapsto df(X) := X(f), \quad p \mapsto df_p(X_p) := X_p(f).$$

For each chart (U, φ), writing $\varphi(p) = (\varphi^1(p), ..., \varphi^n(p))$ just as before gives us that each $\omega \in \Lambda^1_M(U)$ uniquely decomposes as $\omega = \sum_i f_i d\varphi^i$, where f_i are smooth on U.

We proceed to define the wedge product.

Definition 7.19. Let $\omega \in \Lambda^k_M(U)$ be a differential k-form and $\omega' \in \Lambda^l_M(U)$ be a differential l-form. Their wedge product is defined as

$$\omega \wedge \omega' : \quad p \mapsto \omega_p \wedge \omega'_p.$$

Equivalently, we can introduce it by generalising the calculus of the preceding subsection to $C^\infty(U)$-multilinear functions $T : \mathcal{T}_M(U)^m \to C^\infty_M(U)$. For such a T, we define $\sigma_* T$ by setting $\sigma_* T(X_1, ..., X_m) = (-1)^\sigma T(X_{\sigma(1)}, ..., X_{\sigma(m)})$ and then introduce $\mathsf{Alt}(T) := 1/m! \sum_\sigma \sigma_* T$. We can then define

$$\omega \wedge \omega' := \frac{(k+l)!}{k!l!} \mathsf{Alt}(\omega \otimes \omega')$$

Evaluated at each point, the latter definition reproduces the former.

Lemma 7.20. *The wedge product satisfies bilinearity, associativity and graded commutativity properties similar to those of Lemma 7.7. In addition to that, for each $g \in C^\infty(U) = \Lambda^0_M(U)$, we have*

$$(g\omega) \wedge \omega' = g \cdot (\omega \wedge \omega') = \omega \wedge (g\omega'). \ \square$$

Example 7.21. Let $\varphi : U \xrightarrow{\sim} \Omega$ be a chart of a manifold of dimension n. Then for each $\mathcal{I} \in \langle n \rangle^k$ we can take $d\varphi^{\wedge \mathcal{I}} = d\varphi^{i_1} \wedge ... \wedge d\varphi^{i_k} \in \otimes^k \mathcal{T}^*(U)$. If $\mathcal{I}$ is ordered, meaning $1 \le i_1 < ... < i_k \le n$, then for any other ordered $\mathcal{J} \in \langle n \rangle^k$,

$$d\varphi^{\wedge \mathcal{I}}(e^\varphi[\mathcal{J}]) = d\varphi^{i_1} \wedge ... \wedge d\varphi^{i_k}(e^\varphi_{j_1}, ..., e^\varphi_{j_k}) = \delta^{\mathcal{I}}_{\mathcal{J}};$$

Because of this, by repeating the same arguments as around Proposition 7.12, any $\omega \in \Lambda^k_M(U)$ can be uniquely written as

$$\omega = \sum_{\mathcal{J},<} \omega(e^\varphi[\mathcal{J}]) d\varphi^{\wedge \mathcal{J}}$$

where in the sum, we only take ordered multi-indices. If we want, we can also write it as

$$\omega = \frac{1}{k!} \sum_{j_1, ..., j_k} \omega(e^\varphi_{j_1}, ..., e^\varphi_{j_k}) d\varphi^{j_1} \wedge ... \wedge d\varphi^{j_k}.$$

In short, the k-forms on a chart are expressions $\sum_{\mathcal{J}} f_{\mathcal{J}} d\varphi^{\wedge \mathcal{J}}$ with $f_{\mathcal{J}} \in C^\infty(U)$.

We can in particular write $U = \Omega$ and $\varphi = \mathsf{id}$. Then, the differential k-forms on Ω are simply the expressions $\sum_{\mathcal{J}} f_{\mathcal{J}} dx^{\wedge \mathcal{J}}$ with $f_{\mathcal{J}}$ smooth on Ω.

Corollary 7.22. *Let $\omega : p \mapsto \omega_p \in \wedge^k T^*_p M$ be a family of k-forms. Then it is smooth iff for each $p \in M$ there exists a chart (U, φ) containing p such that $\omega|_U = \sum_{\mathcal{J}} f_{\mathcal{J}} d\varphi^{\wedge \mathcal{J}}$ with $f_{\mathcal{J}}$ smooth on U.*

Pullback

Let $F : M \to N$ be a smooth map. Each differential k-form ω on N is in particular a k-tensor, so we can consider its pullback $F^*\omega \equiv F^*(\omega)$. To remind, for each $p \in M$,

$$F^*(\omega)_p((X_1)_p, ..., (X_k)_p) := \omega_{F(p)}(F_*(p)(X_1)_p, ..., F_*(p)(X_k)_p).$$

Here, $(X_i)_p \in T_pM$ and $F_*(p) : T_pM \to T_{F(p)}N$ is the pushforward map.

Proposition 7.23. *In the situation above, one has $F^*G^* = (G \circ F)^*$, and*

1. *For each $f \in C^\infty(N)$, $F^*df = d(f \circ F)$,*
2. *The pullback $F^*\omega$ is a differential k-form on M for any $\omega \in \Lambda^k_N(N)$,*
3. *The pullback operation is linear is compatible with multiplication of forms by functions:*

$$F^*(f\omega + g\omega') = F^*(f) \cdot F^*\omega + F^*(g) \cdot F^*\omega'.$$

More generally, for any two differential forms ω_1, ω_2 on N, one has $F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2$.

Proof. Proposition 6.23 ensures smoothness and Lemma 7.10 checks alternation-related properties at each point. $\square$

Example 7.24. Let us consider a map $F : \Omega \to \Theta$. Denote $(x^1, ..., x^n)$ the coordinates in Ω and $(y^1, ..., y^m)$ the coordinates in Θ. If we write $F(x) = (F^1(x), ..., F^m(x))$, then $y^i \circ F = F^i$, and so $F^*(dy^i) = d(y^i \circ F) = dF^i$.

We already computed that at $p = (x^1, ..., x^n)$, one has $dF^i_p = \sum \frac{\partial F^i}{\partial x^j}(p)dx^j_p$, or simply $dF^i = \sum_j \partial_{x^j} F^i \cdot dx^j$. For each $\mathcal{I} \in \langle m\rangle^k$, we thus get

$$\begin{aligned} F^*(dy^{\wedge\mathcal{I}}) &= dF^{\wedge\mathcal{I}} = dF^{i_1} \wedge ... \wedge dF^{i_k} \\ &= \sum_{j_1,\cdots,j_k} \frac{\partial F^{i_1}}{\partial x^{j_i}} \cdot ... \cdot \frac{\partial F^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge ... \wedge dx^{j_k} \\ &= \sum_{\mathcal{J}\in\langle n\rangle^k} J(F)^{\mathcal{I}}_{\mathcal{J}} dx^{\wedge\mathcal{J}}, \quad J(F)^{\mathcal{I}}_{\mathcal{J}} = \frac{\partial F^{i_1}}{\partial x^{j_i}} \cdot ... \cdot \frac{\partial F^{i_k}}{\partial x^{j_k}} \end{aligned}$$

As a consequence, given any $\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} dy^{\otimes\mathcal{I}}$ with $\omega_{\mathcal{I}}$ smooth functions on Θ (matrix k-tensors attached to each point of Θ), we have

$$F^*\omega = \sum_{\mathcal{J}\in\langle n\rangle^k} \sum_{\mathcal{I}\in\langle m\rangle^k,<} F^*(\omega_{\mathcal{I}}) \cdot J(F)^{\mathcal{I}}_{\mathcal{J}} \cdot dx^{\wedge\mathcal{J}}.$$

Note that while the $\mathcal{I}$-sum is over the ordered indices, the $\mathcal{J}$-sum is not.

One particular case is when $F : \Omega \to \Omega$ and we consider the top differential form $dx^1 \wedge ... \wedge dx^n$. Lemma 7.14 implies that

$$F^*(dx^1 \wedge ... \wedge dx^n) = \det J(F) \cdot dx^1 \wedge ... \wedge dx^n,$$

where at each $p \in \Omega$, $\det J(F)(p)$ is the determinant of the Jacobian matrix $J(F)^i_j(p)$.

7.3 De Rham differential

The assignment $f \mapsto df$ is a linear map $d : \Lambda^0_M(U) = C^\infty_M(U) \to \Lambda^1_M(U)$. One of the reasons for interest in differential forms is that they usefully extend the differential map.

Theorem 7.25. *Let M be a smooth manifold. For each $U \in \mathsf{Op}\, M$ and $k \geq 0$, there is a map*

$$d \equiv d^{(k)} : \Lambda^k_M(U) \to \Lambda^{k+1}_M(U), \quad \omega \mapsto d\omega \equiv d(\omega) \equiv d^k(\omega),$$

called the de Rham differential, that satisfies the following:

1. $d^0(f) = df$.
2. *Each $d^{(k)}$ is linear and $d^2 = d^{(k+1)} \circ d^{(k)} = 0$.*
3. *For $\omega \in \Lambda^k_M(U)$, $\eta \in \Lambda^l_M(U)$, the differential satisfies the graded Leibniz rule*

$$d^{(k+l)}(\omega \wedge \eta) = d^{(k)}(\omega) \wedge \eta + (-1)^k \omega \wedge d^{(l)}(\eta).$$

4. *For each open subset $V \subset U$ we have $(d\omega)|_V = d(\omega|_V)$.*

Subject to these conditions, d is furthermore unique.

In particular, if (U, φ) is a chart and $\omega = \sum \alpha_{\mathcal{I}} d\varphi^{\wedge\mathcal{I}}$ with $\alpha_{\mathcal{I}}$ smooth, then $d\omega = \sum d\alpha_{\mathcal{I}} \wedge d\varphi^{\wedge\mathcal{I}}$.

Remark 7.26. The condition 4. can be reformulated, as usual, via the requirement that the following diagram commutes:

$$\begin{tikzcd} \Lambda^k_M(U) \arrow[r, "d"] \arrow[d, "|_V"'] & \Lambda^{k+1}_M(U) \arrow[d, "|_V"] \\ \Lambda^k_M(V) \arrow[r, "d"] & \Lambda^{k+1}_M(V) \end{tikzcd}$$

It is in fact a non-condition, implied by the other requirements 1. − 3., but I find it easier to include it explicitly. It suggests that just like for the differential, the value $d\omega_p$ only depends on a small neighbourhood of p.

As I tried to show above, it is custom to suppress the indices of the differentials, and write

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

The formula for d in a chart is similar for each k-form, which explains such an ambiguity in notation.

Let us convince ourselves that the theorem is true by computing some examples.

De Rham differential on $\mathbb{R}^n$

Example 7.27. Take $M = I \subset \mathbb{R}$. Then the only interesting operation is $f \mapsto df$, $df_t = f'(t)dt$. As we know there are no 2-forms here, so $d^2 f = 0$ always.

Example 7.28. Take $M = \Omega \subset \mathbb{R}^2$. Then $\Lambda^1(\Omega)$ is C^∞-spanned by dx, dy, and $\Lambda^2(\Omega)$ – by $dx \wedge dy$. This means that any 1-form is written as $\eta = Adx + Bdy$ and any 2-form is $\omega = Cdx \wedge dy$ where $A, B, C \in C^\infty(\Omega)$.

We know that for $f \in C^\infty(\Omega)$, $df = \partial_x f\, dx + \partial_y f\, dy$. Thus $dA = \partial_x A\, dx + \partial_y A\, dy$ and similarly for B. We compute

$$d\eta = d(Adx + Bdy) = dA \wedge dx + dB \wedge dy,$$

and since $dx \wedge dx = dy \wedge dy = 0$ we only care about $\partial_y A dy$ and $\partial_x B dx$. We continue

$$d\eta = \partial_y A\, dy \wedge dx + \partial_x B\, dx \wedge dy = (\partial_x B - \partial_y A) dx \wedge dy.$$

Now, if $\eta = df$ then $A = \partial_x f$ and $B = \partial_y f$, and thus

$$\partial_x B - \partial_y A = \partial_x \partial_y f - \partial_y \partial_x f = 0,$$

meaning $d^2 f = 0$. Finally, for dimension reasons, $d\omega$ is always zero.

Example 7.29. Take $M = \Omega \subset \mathbb{R}^3$. Then

1. $\Lambda^1(\Omega)$ is C^∞-spanned by dx, dy, dz,
2. $\Lambda^2(\Omega)$ – by $dx \wedge dy$, $dx \wedge dz$, $dy \wedge dz$,
3. $\Lambda^3(\Omega)$ – by $dx \wedge dy \wedge dz$.

As before, $df = \partial_x f dx + \partial_y f dy + \partial_z f dz$. Now, let $\eta = Adx + Bdy + Cdz$. We are to compute $d\eta = dA \wedge dx + dB \wedge dy + dC \wedge dz$. We find:

$$\begin{aligned} d\eta &= (\partial_x A\, dx + \partial_y A\, dy + \partial_z A\, dz) \wedge dx + (\partial_x B\, dx + \partial_y B\, dy + \partial_z B\, dz) \wedge dy \\ &\quad + (\partial_x C\, dx + \partial_y C\, dy + \partial_z C\, dz) \wedge dz \\ &= (\partial_y A\, dy + \partial_z A\, dz) \wedge dx + (\partial_x B\, dx + \partial_z B\, dz) \wedge dy + (\partial_x C\, dx + \partial_y C\, dy) \wedge dz \\ &= (\partial_x B - \partial_y A) dx \wedge dy + (\partial_x C - \partial_z A) dx \wedge dz + (\partial_y C - \partial_z B) dy \wedge dz. \end{aligned}$$

Perhaps some of you are familiar with the notion of curl. If we consider the vector field $F = A\partial_x + B\partial_y + C\partial_z$, then its curl, denoted $\nabla \times F$, is given by

$$\nabla \times F = (\partial_y C - \partial_z B)\, \partial_x + (\partial_z A - \partial_x C)\, \partial_y + (\partial_x B - \partial_y A)\partial_z.$$

Thus the coefficients of $d\eta$ are the same, up to a sign and order, as the coefficients of $\nabla \times F$. This is not an artefact, and there are ways to make this connection precise.

We continue by computing $d\omega$ for $\omega = Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$:

$$\begin{aligned} d\omega &= dP \wedge dx \wedge dy + dQ \wedge dx \wedge dz + dR \wedge dy \wedge dz \\ &= \partial_z P\, dz \wedge dx \wedge dy + \partial_y Q\, dy \wedge dx \wedge dz + \partial_x R\, dx \wedge dy \wedge dz \\ &= (\partial_x R - \partial_y Q + \partial_z P) dx \wedge dy \wedge dz. \end{aligned}$$

Now let us plug in $\omega = d\eta$:

$$\begin{aligned} d^2\eta &= (\partial_x\partial_y C - \partial_x\partial_z B - \partial_y\partial_x C + \partial_y\partial_z A \\ &\quad + \partial_z\partial_x B - \partial_z\partial_y A) dx \wedge dy \wedge dz \\ &= 0. \end{aligned}$$

If we have a vector field $V = V^1\partial_x + V^2\partial_y + V^3\partial_z$, then its divergence is defined as $div(V) = \partial_x V^1 + \partial_y V^2 + \partial_z V^3$. Taking into account signs and orders, the identity $d^2\eta = 0$ can be written as $div(\nabla \times F) = 0$.

Another observation that I leave to compute by hand is that $d^2 f = 0$. Note that $df = \partial_x f\, dx + \partial_x f\, dy + \partial_x f\, dz$ has as coefficients the functions of ∇f, the gradient of f. Thus $\nabla \times (\nabla f) = 0$.

De Rham differential, locally

Proposition 7.30. *The De Rham differential exists and is unique for any $\Omega \subset \mathbb{R}^n$.*

Proof. It takes a while, but everything is very natural.

1. Each $\omega \in \Lambda^k(\Omega)$ *uniquely* decomposes as

$$\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} dx^{\wedge\mathcal{I}}, \quad \mathcal{I} = (i_1, \dots, i_k), \quad 1 \le i_1 < \dots < i_k \le n.$$

   This can be read as sums of products of 0-forms (functions) ω_I and k-forms $dx^{\wedge\mathcal{I}}$. For each $dx^{i_1} \wedge \dots \wedge dx^{i_k}$, since $dx^j = d(x^j)$, $d^2 = 0$ and Leibniz rule, we get $d(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = 0$. The same Leibniz rule insists that if d exists, it *must* work as $f \mapsto df$ and

$$d\omega = \sum_{\mathcal{I},<} d\omega_{\mathcal{I}} \wedge dx^{\wedge\mathcal{I}}$$

   where $d\omega_{\mathcal{I}}(X) = X(\omega_{\mathcal{I}})$ as usual. This covers the uniqueness part, but we need to work out that d defined via the formula above indeed satisfies $1. - 4$.

2. The assignment $\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} dx^{\wedge\mathcal{I}} \mapsto \sum_{\mathcal{I},<} d\omega_{\mathcal{I}} \wedge dx^{\wedge\mathcal{I}} = d\omega$ is $\mathbb{R}$-linear: we use that $f \mapsto df$ is $\mathbb{R}$-linear together with the fact that

$$\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} dx^{\wedge\mathcal{I}}, \omega' = \sum_{\mathcal{I},<} \omega'_{\mathcal{I}} dx^{\wedge\mathcal{I}}, \quad a\omega + b\omega' = \sum_{\mathcal{I},<} (a\omega_{\mathcal{I}} + b\omega'_{\mathcal{I}}) dx^{\wedge\mathcal{I}}.$$

3. If we have a form $A = \alpha dx^{\wedge\mathcal{I}}$ with $\mathcal{I}$ ordered, then the definition above means that $dA = d\alpha \wedge dx^{\wedge\mathcal{I}}$. What about arbitrary $\mathcal{I}$?

   If $\mathcal{I}$ has a repeating index, then $A = \alpha dx^{\wedge\mathcal{I}} = 0$, and $d\alpha \wedge dx^{\wedge\mathcal{I}} = d\alpha \wedge 0 = 0$.

   If $\mathcal{I}$ has no repeating indices, then there is unique ordered $\mathcal{J}$ and a permutation $\sigma \in \Sigma_k$ taking $\mathcal{I}$ to $\mathcal{J}$, so that

$$A = \alpha dx^{\wedge\mathcal{I}} = (-1)^\sigma \alpha dx^{\wedge\mathcal{J}}$$

   the latter is a presentation in the basis, so by definition above,

$$dA = d((-1)^\sigma\alpha) \wedge dx^{\wedge\mathcal{J}} = (-1)^\sigma d\alpha \wedge dx^{\wedge\mathcal{J}} = d\alpha \wedge ((-1)^\sigma dx^{\wedge\mathcal{J}}) = d\alpha \wedge dx^{\wedge\mathcal{I}}.$$

   The conclusion is that for any sum $A = \sum_{\mathcal{I}} A_{\mathcal{I}} dx^{\wedge\mathcal{I}}$ (not necessarily over ordered multi-indices $\mathcal{I}$), we have $dA = \sum_{\mathcal{I}} dA_{\mathcal{I}} \wedge dx^{\wedge\mathcal{I}}$.

4. Let us now prove that $d^2 = 0$. For $\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} dx^{\wedge\mathcal{I}}$, we have

$$d\omega = \sum_j \sum_{\mathcal{I},<} \frac{\partial \omega_{\mathcal{I}}}{\partial x^j} dx^j \wedge dx^{\wedge\mathcal{I}}$$

this double sum can be regarded as a sum over some range of $k+1$-multiindices $(j, i_1, ..., i_k)$, not necessarily ordered. We previously established that the formula for d does not care about that, so

$$\begin{aligned} d^2\omega &= \sum_j \sum_{\mathcal{I},<} d\left(\frac{\partial \omega_{\mathcal{I}}}{\partial x^j}\right) \wedge dx^j \wedge dx^{\wedge\mathcal{I}} \\ &= \sum_{j,l} \sum_{\mathcal{I},<} \left(\frac{\partial^2 \omega_{\mathcal{I}}}{\partial x^l \partial x^j}\right) dx^l \wedge dx^j \wedge dx^{\wedge\mathcal{I}} \\ &= \sum_{j,l} \sum_{\mathcal{I},<} \frac{1}{2}\left(\frac{\partial^2 \omega_{\mathcal{I}}}{\partial x^l \partial x^j} - \frac{\partial^2 \omega_{\mathcal{I}}}{\partial x^j \partial x^l}\right) dx^l \wedge dx^j \wedge dx^{\wedge\mathcal{I}} = 0. \end{aligned}$$

The following trick was applied here: present the sum as two times the sume of half parts, relabel indices $j \leftrightarrow l$ and use $dx^j \wedge dx^l = -dx^l \wedge dx^j$. The commutativity of second-order derivatives then does the rest.

5. We now prove the graded Leibniz rule. For functions $f, g \in \Lambda^0(\Omega)$, we know (PS5) that $d(fg) = gdf + fdg$. Let $\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} dx^{\wedge\mathcal{I}}$, $\eta = \sum_{\mathcal{J},<} \eta_{\mathcal{J}} dx^{\wedge\mathcal{J}}$ where $\mathcal{I}$ is a k-multiindex and $\mathcal{J}$ is an l-multiindex. Then by Lemma 7.20,

$$\omega \wedge \eta = \sum_{\mathcal{I},<} \sum_{\mathcal{J},<} (\omega_{\mathcal{I}} \cdot \eta_{\mathcal{J}}) dx^{\wedge\mathcal{I}} \wedge dx^{\wedge\mathcal{J}}.$$

Thus

$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{\mathcal{I},<} \sum_{\mathcal{J},<} d(\omega_{\mathcal{I}} \cdot \eta_{\mathcal{J}}) \wedge dx^{\wedge\mathcal{I}} \wedge dx^{\wedge\mathcal{J}} \\ &= \sum_{\mathcal{I},<} \sum_{\mathcal{J},<} (\eta_{\mathcal{J}} d\omega_{\mathcal{I}} + \omega_{\mathcal{I}} d\eta_{\mathcal{J}}) \wedge dx^{\wedge\mathcal{I}} \wedge dx^{\wedge\mathcal{J}} \\ &= \sum_{\mathcal{I},<} \sum_{\mathcal{J},<} [(d\omega_{\mathcal{I}} \wedge dx^{\wedge\mathcal{I}}) \wedge (\eta_{\mathcal{J}} \wedge dx^{\wedge\mathcal{J}}) \\ &+ (-1)^k (\omega_{\mathcal{I}} \wedge dx^{\wedge\mathcal{I}}) \wedge (d\eta_{\mathcal{J}} \wedge dx^{\wedge\mathcal{J}})] \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

6. We have shown that for each open $\Omega \subset \mathbb{R}^n$, we have $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ satisfying $1.-3.$ of Theorem 7.25, and that such a d is unique. Its formula, for each $\omega \in \Lambda^k(\Omega)$ and $p \in \Omega$, is given by

$$(d\omega)_p = \sum_{\mathcal{I},<} (d\omega_{\mathcal{I}})_p \wedge dx_p^{\wedge\mathcal{I}}.$$

This expression, eventually, works by computing differentials of the component functions $\omega_{\mathcal{I}}$, and thus only depends on some small neighbourhood of p. We thus have the property 4.: for each $U \subset \Omega$, $d\omega|_U$ coincides with $d(\omega|_U)$ (here we use that U is also open in $\mathbb{R}^n$). $\square$

Lemma 7.31. *Let $\Omega \in \mathsf{Op}\,\mathbb{R}^n$, $\Theta \in \mathsf{Op}\,\mathbb{R}^m$ and $F : \Omega \to \Theta$ smooth. Then for each $\omega \in \Lambda^k(\Theta)$, $F^* d\omega = d(F^*\omega)$.*

Proof. For $\omega = \sum \omega_{\mathcal{I}} dy^{\wedge \mathcal{I}}$, denoting $F = (F^1, ..., F^m)$, we have

$$F^* d\omega = F^* \left(\sum d\omega_{\mathcal{I}} \wedge dy^{\wedge \mathcal{I}}\right) = \sum d(\omega_{\mathcal{I}} \circ F) \wedge dF^{\wedge \mathcal{I}},$$

$$d(F^*\omega) = d \sum (\omega_{\mathcal{I}} \circ F) \wedge dF^{\wedge \mathcal{I}} = \sum d(\omega_{\mathcal{I}} \circ F) \wedge dF^{\wedge \mathcal{I}}.$$

They are the same picture. □

Proof of Theorem 7.25

We go through the steps that yield De Rham differential on a general manifold M:

1. For a chart $\varphi : U \xrightarrow{\sim} \Omega$ and $\omega \in \Lambda^k(U)$, we define $d\omega := \varphi^*(d((\varphi^{-1})^*\omega))$. In terms of basis decomposition, observe that

$$\omega = \sum_{\mathcal{I},<} \omega_{\mathcal{I}} d\varphi^{\wedge \mathcal{I}}, \quad (\varphi^{-1})^*\omega = \sum_{\mathcal{I},<} ((\varphi^{-1})^*\omega_{\mathcal{I}}) dx^{\wedge \mathcal{I}},$$

$$d((\varphi^{-1})^*\omega) = \sum_{\mathcal{I},<} d((\varphi^{-1})^*\omega_{\mathcal{I}}) \wedge dx^{\wedge \mathcal{I}},$$

$$\varphi^*(d((\varphi^{-1})^*\omega)) = \sum_{\mathcal{I},<} \varphi^* d((\varphi^{-1})^*\omega_{\mathcal{I}}) \wedge dx^{\wedge \mathcal{I}}.$$

   For the differentials of 0-forms a.k.a. functions we know that $F^* df = d(F^* f)$, so $\varphi^* d((\varphi^{-1})^*\omega_{\mathcal{I}}) = d(\varphi^*(\varphi^{-1})^*\omega_{\mathcal{I}}) = d\omega_{\mathcal{I}}$. Thus

$$d\omega = \sum_{\mathcal{I},<} d\omega_{\mathcal{I}} \wedge d\varphi^{\wedge \mathcal{I}}.$$

   We can repeat the argument for any $W \subset U$ and $\varphi|_W$, getting that $d\omega|_W = d(\omega|_W) = \sum_{\mathcal{I},<} d\omega_{\mathcal{I}}|_W \wedge d\varphi^{\wedge \mathcal{I}}|_W$.

2. A check using Lemma 7.31 shows that nothing depends on the choice of the chart map φ:

$$\begin{aligned}\varphi^*(d((\varphi^{-1})^*\omega)) &= \psi^*(\psi^{-1})^*\varphi^*(d((\varphi^{-1})^*\omega)) = \psi^*(\varphi \circ \psi^{-1})^*(d((\varphi^{-1})^*\omega)) \\ &= \psi^* d((\varphi \circ \psi^{-1})^*(\varphi^{-1})^*\omega) = \psi^* d((\psi^{-1})^*\varphi^*(\varphi^{-1})^*\omega)) \\ &= \psi^* d((\psi^{-1})^*\omega).\end{aligned}$$

   This whole check uses that $\varphi \circ \psi^{-1}$ is a map between opens in Euclidean spaces.

3. Then we can verify all the properties 1. − 3. by passing back and forth between U and Ω. In particular, for any linear combination $\sum \alpha_{\mathcal{I}} df^{\wedge \mathcal{I}}$, where f^i are arbitrary functions on U, we have $d \sum \alpha_{\mathcal{I}} df^{\wedge \mathcal{I}} = \sum d\alpha_{\mathcal{I}} \wedge df^{\wedge \mathcal{I}}$. Everything is similar for $W \subset U$, so we have 4..

4. To generalise to the whole manifold M, choose a cover by charts U_i. For each U_i, we have that d exists and is unique. Given $\omega \in \Lambda^k(U_i \cup U_j)$, the uniqueness implies that

$$d(\omega|_{U_i})|_{U_i \cap U_j} = d(\omega|_{U_i \cap U_j}) = d(\omega|_{U_j})|_{U_i \cap U_j}.$$

   For this reason, given $\omega \in \Lambda^k(M)$, it is nautral to define $d\omega$ to be the unique $k+1$-form such that $d\omega|_{U_i} = d(\omega|_{U_i})$. Sheaf property of forms will imply its existence. The properties 1. − 3. are then verified on each U_i.

5. Finally, the same arguments can be issued for M replaced by $V \in \mathsf{Op}\, M$, and U_i by $U_i \cap V$. The uniqueness of d implies that locally, it looks always like $d \sum \alpha_{\mathcal{I}} df^{\wedge \mathcal{I}} = \sum d\alpha_{\mathcal{I}} \wedge df^{\wedge \mathcal{I}}$, so we have 4. □

Lemma 7.32. *Let $F : M \to N$ be smooth. Then for each $\omega \in \Lambda^k(N)$, $F^* d\omega = d(F^*\omega)$.*

Proof. True locally, use sheaves. □

De Rham cohomology

The identity $d^2 = 0$ implies that $\ker(d : \Lambda^k(M) \to \Lambda^{k+1}(M))$ contains $\mathrm{im}(d : \Lambda^{k-1}(M) \to \Lambda^k(M))$.

Definition 7.33. Let M be a manifold.

1. A form $\omega \in \Lambda^k(M)$ is *closed* if $d\omega = 0$,
2. A form $\omega \in \Lambda^k(M)$ is *exact* if $\omega = d\eta$, $\eta \in \Lambda^{k-1}(M)$.
3. The k-th *de Rham cohomology group* $H^k_{dR}(M)$ is defined as the quotient of the set of all closed k-forms by the set of all exact k-forms: $\omega \sim \omega'$ iff $\omega - \omega' = d\eta$.

In other words, $H^k_{dR}(M) = \frac{\ker(d:\Lambda^k(M)\to\Lambda^{k+1}(M))}{\mathrm{im}(d:\Lambda^{k-1}(M)\to\Lambda^k(M))}$. Not only it is an abelian group, it is a vector space. But what is it, really?

Proposition 7.34. *(Poincaré's lemma) For $M = \mathbb{R}^n$, one has $H^0_{dR}(\mathbb{R}^n) \cong \mathbb{R}, H^{\geq 1}_{dR}(\mathbb{R}^n) = 0$, meaning that any closed $k \geq 1$ form is exact. More generally, we can replace $\mathbb{R}^n$ by any contractible (for example, starlike) subset of $\mathbb{R}^n$.*

It is not true that for an arbitrary open $\Omega \subset \mathbb{R}^n$ we have the same statement. For instance one can show that $H^1_{dR}(\mathbb{R}^2 \setminus 0) = \mathbb{R}$, with the closed but not exact form given by $\frac{1}{x^2+y^2}(xdy - ydx)$.

Example 7.35.

1. For $\mathbb{S}^n$, one has $H^0_{dR}(\mathbb{S}^n) \cong H^n_{dR}(\mathbb{S}^n) \cong \mathbb{R}$ and $H^k_{dR}(\mathbb{S}^n) = 0$ otherwise. The closed-but-not-exact form giving nonzero H^n is the volume form of the sphere, discussed in TD and further below.
2. For $\mathbb{T}$, one has $H^0_{dR}(\mathbb{T}) \cong H^2_{dR}(\mathbb{T}) \cong \mathbb{R}$, $H^1_{dR}(\mathbb{T}) \cong \mathbb{R}^2$. Denoting $q : \{(x,y)\} = \mathbb{R}^2 \to \mathbb{T}$ the quotient map, one can show that there are two 1-forms ω_1, ω_2 on $\mathbb{T}$ such that $q^*\omega_1 = dx, q^*\omega_2 = dy$.

   Since q is a local diffeomorphism, one can conclude that $d\omega_1 = d\omega_2 = 0$, but these forms will not be exact. One then shows that $H^1_{dR}(\mathbb{T}) = \mathsf{Span}([\omega_1],[\omega_2]), H^2_{dR}(\mathbb{T}) = \mathsf{Span}([\omega_1 \wedge \omega_2])$.
3. For real projective spaces, $H^0_{dR}(\mathbb{RP}^n) \cong \mathbb{R}$, $H^n_{dR}(\mathbb{RP}^n) \cong \mathbb{R}$ if n is odd, $H^n_{dR}(\mathbb{RP}^n) = 0$ if n is even, and is zero otherwise.

The famous de Rham theorem constructs an isomorphism between de Rham cohomology and singular cohomology of manifolds.

8 Orientation, integration, Stokes

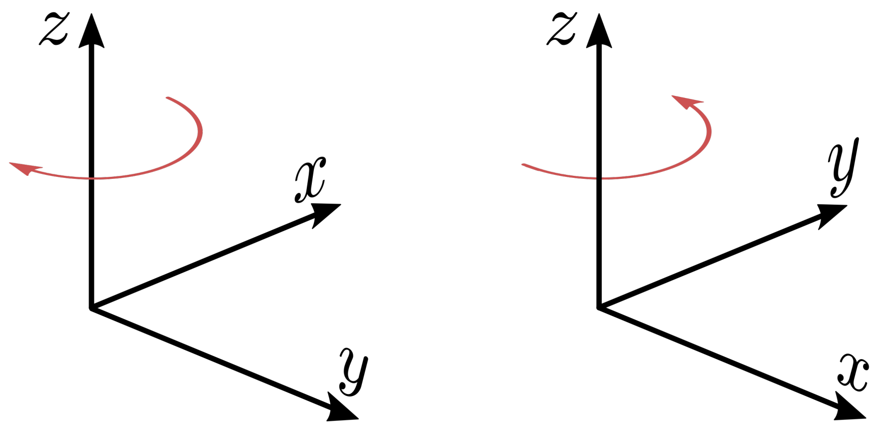
Orientation of $\mathbb{R}$-vector spaces

Let V be a $\mathbb{R}$-vector space of dimension n.

When $n = 2$, we have an intuitive notion of understanding of its orientation. Envisioning V as a plane, we can speak of "clockwise" or "counterclockwise" orientation.

When $n = 3$, we also have an intuitive notion of orientation. If we fix a plane $H \subset V$, we can orient it as before, say counterclockwise. Then the question will be where to point the orthogonal direction, upstairs or downstairs. Or, alternatively, we could fix the orthogonal direction, and play with the orientation of H. This produces two choices, left-handed and right-handed.

In general, we can handle orientation using linear transformations.



Source: wikipedia

We recall from PS1 that $\mathsf{GL}_n(\mathbb{R})$ has two connected components, characterised by the sign of the determinant. The positive determinant component shall be denoted $\mathsf{GL}_n^+(\mathbb{R})$. Below, we also work with ordered bases of V, that we can understand as tuples $(e_1, \dots, e_n) \in V^n$.

Definition 8.1. Two ordered bases $(e_1, \dots, e_n)$ and $(\epsilon_1, \dots, \epsilon_n)$ of V define the same orientation if the linear map F defined as $F(e_i) = \epsilon_i$ has positive determinant. This is seen to be an equivalence relation.

An *orientation* of V is an equivalence class of ordered bases, with respect to the relation described above.

Given any two ordered bases e_i and ϵ_i, the map $F(e_i) = \epsilon_i$ is always invertible, so its determinant is either positive or negative. Thus there are two orientations possible on each V.

Lemma 8.2. *Let $\omega \in \Lambda^n V^*$ be a nonzero n-form. Then e_i and ϵ_i define the same orientation iff $\omega(e_1, ..., e_n)$ and $\omega(\epsilon_1, ..., \epsilon_n)$ are of the same sign.*

Proof. Lemma 7.14 implies that for any linear $F : V \to V$, $F^*\omega = \det F \cdot \omega$. For $F(e_i) = \epsilon_i$, we note that

$$\omega(\epsilon_1, ..., \epsilon_n) = \omega(F(e_1), ..., F(e_n)) = F^*\omega(e_1, ..., e_n) = \det F \cdot \omega(e_1, ..., e_n).$$

We then conclude since $\det F > 0$ for equivalent bases. □

Consequently, we see that choosing an orientation is the same as choosing a nonzero $\omega \in \Lambda^n V^*$. Indeed, we can then consider all the bases $e_1, ..., e_n$ such that $\omega(e_1, ..., e_n) > 0$.

Differently put, given a basis $e_1, ..., e_n$, we can consider the dual basis $f^1, ..., f^n$. The form $\omega = f^1 \wedge ... \wedge f^n$ will then yield positive values on all bases of the same orientation as e_i, and $\omega = -f^1 \wedge ... \wedge f^n$ will correspond to the opposite orientation.

Multiplying these two forms by a positive number does not change the result. Conclusion: orientations on V correspond to connected components of $\Lambda^n V^* \setminus 0 \cong \mathbb{R} \setminus 0$.

Orientation of manifolds

In the context of manifolds, we are met with the issue of variety. A differential form on M assigns a linear n-form at each point of M, and we could try saying that perhaps we simply require that all these forms are "of the same sign". But in any case we would perhaps like to have some other characterisation of orientation.

First, we generalise the linear setup to opens in $\mathbb{R}^n$. Let $F : \Omega \xrightarrow{\sim} \Theta$ be a diffeomorphism between two such opens.

Definition 8.3. F is called *orientation-preserving* if $\det J(F)$, the determinant of the Jacobian of F, is positive at each point of Ω. Otherwise F is called *orientation-reversing.*

Here, we use the tangent space map $F_*(p) : T_p\Omega \to T_{F(p)}\Theta$ to generalise our idea of orientation.

Example 8.4. For $\Omega = \{(x, y) | x > 0, y > 0\}$, we can consider the diffeomorphism $F : (x, y) \mapsto (x^2, y^2)$. We see that

$$F_* : \left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p \mapsto 2x\left.\frac{\partial}{\partial x}\right|_{F(p)}, 2y\left.\frac{\partial}{\partial y}\right|_{F(p)}.$$

Since $x, y \neq 0$ we have that the resulting tangent vectors are a basis. If we identify each $T_q\Omega$ with $\mathbb{R}^2$, $v^1\partial_x|_q + v^2\partial_y|_q \mapsto (v^1, v^2)$, then the two bases written above become $(1, 0), (0, 1)$ and $(2x, 0), (0, 2y)$ and they define the same orientation.

Considering $G(x, y) = (y^2, x^2)$ will, on the other hand, reverse the orientation of tangent spaces.

Many authors further develop a natural generalisation of an ordered basis to this context: they consider ordered families of nowhere vanishing vector fields. We shall rather focus on forms.

Lemma 8.5. *Let $F : \Omega \xrightarrow{\sim} \Theta$ be an orientation-preserving diffeomorphism of opens in $\mathbb{R}^n = (x^1, ..., x^n)$. Denoting $\omega = dx^1 \wedge ... \wedge dx^n \in \Lambda^n(\Theta)$, we have $F^*\omega = f \cdot dx^1 \wedge ... \wedge dx^n \in \Lambda^n(\Omega)$, where f is a positive smooth function on Ω.*

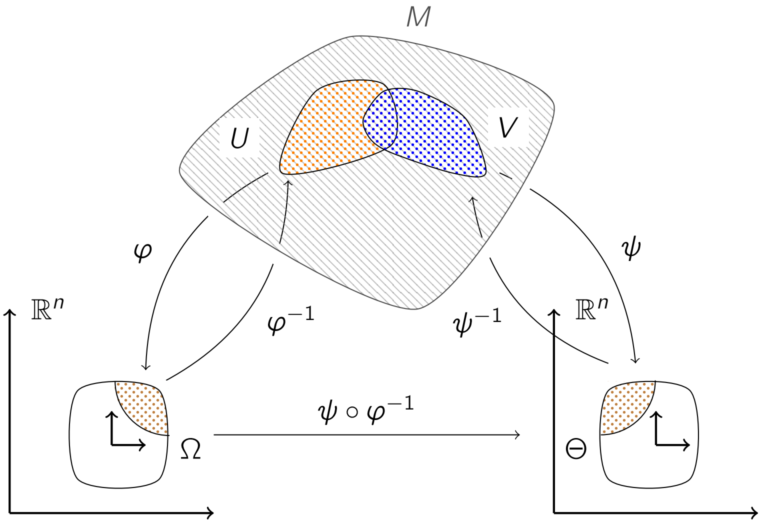
Proof. $F^*(dx^1 \wedge ... \wedge dx^n) = \det J(F) \cdot dx^1 \wedge ... \wedge dx^n$ as per Example 7.24. □

Specifying a form $\omega = f\, dx^1 \wedge ... dx^n$ on Ω with $f > 0$ provides a way to decide, at each $p \in \Omega$, which basis $e_1, ..., e_n$ of $T_p\Omega$ is positively oriented: one simply requires that $\omega_p(e_1, ..., e_n) > 0$.

Definition 8.6. Let M be a smooth manifold. Two charts (U, φ) and (V, ψ) are said to be *orientation compatible* if the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is orientation preserving (note that the same is automatically true for its inverse).

M is called *orientable* if it admits a cover by charts (U_i, φ_i) such that each pair U_i, U_j is orientation-compatible.

I reproduce a familiar picture, but now with some orientations in it.



Proposition 8.7. *Let M is a (paracompact) manifold of dimension n. Then M is orientable iff there exists a nowhere zero $\omega \in \Lambda^n(M)$, called the volume form of M.*

The nowhere zero condition means that for each $p \in M$, the map $\omega_p : T_pM^n \to \mathbb{R}$ is nonzero.

Proof.

1. Assume such a form ω exists. Choose an atlas $\varphi_i : U_i \xrightarrow{\sim} \Omega_i$. We know that ω restricted to each connected component of U_i looks like $f d\varphi_i^1 \wedge ... \wedge d\varphi_i^n$, with f either strictly positive or negative. If $f > 0$, do nothing, if $f < 0$, replace φ_i^1 on that component by $-\varphi_i^1$.

2. Now, for any $U_i \cap U_j$, we have $f d\varphi_i^1 \wedge ... \wedge d\varphi_i^n = g d\varphi_j^1 \wedge ... \wedge d\varphi_j^n$ with $f, g > 0$. Note that because φ_i is a diffeomorphism, we can write $f d\varphi_i^1 \wedge ... \wedge d\varphi_i^n = \varphi_i^*((f \circ \varphi_i^{-1}) \cdot dx^1 \wedge ... \wedge dx^n)$. We thus get

$$\varphi_i^*((f \circ \varphi_i^{-1}) \cdot dx^1 \wedge ... \wedge dx^n) = \varphi_j^*((g \circ \varphi_j^{-1}) \cdot dx^1 \wedge ... \wedge dx^n),$$

and, denoting by $F = \varphi_j \circ \varphi_i^{-1}$, we get

$$\begin{aligned}(f \circ \varphi_i^{-1}) \cdot dx^1 \wedge \dots \wedge dx^n &= F^*((g \circ \varphi_j^{-1}) \cdot dx^1 \wedge \dots \wedge dx^n) \\ &= \det J(F) \cdot F^*(g \circ \varphi_j^{-1}) \cdot dx^1 \wedge \dots \wedge dx^n.\end{aligned}$$

The pullbacks do not change positivity of functions, so we must have $\det J(F) > 0$.

3. Suppose we have an atlas $\varphi_i : U_i \xrightarrow{\sim} \Omega_i$ of orientation-compatible charts. We can take, on each U_i, the form $\omega_i = d\varphi_i^1 \wedge \dots \wedge d\varphi_i^n$. By definition, this form satisfies

$$\omega_i(e_1^{\varphi_i}, \dots, e_n^{\varphi_i}) = 1,$$

where $e_k^{\varphi_i}$ are the standard basis vector fields. Using the chart transition arguments, one can show that on $U_i \cap U_j$,

$$\omega_i(e_1^{\varphi_j}, \dots, e_n^{\varphi_j}) = \det J(\varphi_i \circ \varphi_j^{-1}) > 0$$

for each U_j that intersects U_i.

4. The abstract topological nonsense implies the existence of a partition of unity: there is a set of functions $\{p_i\}$, $p_i \in C^\infty(M)$ such that

   (a) $p_i(x) \in [0, 1]$ for all $x \in M$ and all i,

   (b) $\operatorname{supp} p_i \subset U_i$,

   (c) the set $\{\operatorname{supp} p_i\}$ is locally finite: for each $x \in M$ there is an open $U \ni x$ that intersects only with a finite number of $\operatorname{supp} p_i$,

   (d) $\sum_i p_i(x) = 1$ and is a sum over a finite number of terms for all $x \in M$.

5. One then considers the forms $p_i\omega_i$ (they can be made into global forms on M using the usual sheaf argument) and $\omega = \sum p_i\omega_i$. One shows that this defines an n-form on M. Moreover, if $p \in U_j$, then

$$\omega(e_1^{\varphi_j}, \dots, e_n^{\varphi_j})(p) = \sum_i \det J(\varphi_i \circ \varphi_j^{-1})(p) > 0$$

where we sum, due to the properties of partitions of unity, only over a finite number of terms. □

The existence of nowhere zero n-form can be taken as a definition of orientability. Choosing a particular volume form ω amounts to choosing an orientation of M: by the proof above, we can fix chart maps in such a way that $\omega|_U = f\, d\varphi^1 \wedge \dots \wedge d\varphi^n$ with f positive on the chart U.

Example 8.8. The circle $\mathbb{S}^1$ is orientable. In terms of forms, we can take $\eta = x\, dy - y\, dx$ on $\mathbb{R}^2$ and consider its pullback $\omega = i^*\eta$ by the inclusion map $i : \mathbb{S}^1 \to \mathbb{R}^2$. To see that it is nowhere zero, note that if $\omega_p = 0$, this means that for any $v \in T_p(\mathbb{S}^1)$, we have $\omega_p(v) = \eta_p(i_*(v)) = 0$. However, as we know there is v such that $i_* v = x\, \partial_y|_p - y\, \partial_x|_p$, and so

$$\eta_p(i_*(v)) = (x\, dy_p - y\, dx_p)(x\, \partial_y|_p - y\, \partial_x|_p) = x^2 + y^2|_{(x,y)=p} = 1$$

since we are computing it at p in $\mathbb{S}^1$.

Example 8.9. The sphere $\mathbb{S}^2$ is also orientable. The following form is a natural candidate for the volume form on $\mathbb{S}^2$. consider $\omega = dx \wedge dy \wedge dz$ and the Euler vector field $E = x\partial_x + y\partial_y + z\partial_z$. Define then $\iota_E\omega$ by setting $\iota_E\omega(X, Y) = \omega(E, X, Y)$ for each pair of vector fields X, Y. This gives a 2-form whose expression can be computed to be

$$\iota_E\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy.$$

We then define $\omega_E := i^*\iota_E\omega$, where i is again the inclusion map $\mathbb{S}^2 \to \mathbb{R}^3$. We need to see why this is a nowhere zero 2-form.

At each point $p \in \mathbb{R}^3 \setminus 0$, it is possible to find two tangent vectors V_p, W_p such that

1. $F_*V_p = F_*W_p = 0$, where F is the map $F(x, y, z) = x^2 + y^2 + z^2$,
2. The triple E_p, V_p, W_p satisfies $\omega_p(E_p, V_p, W_p) > 0$.

In geometric terms, we simply take an orthogonal complement to E_p (for the standard Riemannian metric) and take a basis in it that gives a positive orientation of $\mathbb{R}^3$.

Now, when $p \in \mathbb{S}^2$, then V_p, W_p belong to $\ker F_* = \operatorname{im} i_*$, so they can be expressed as $V_p = i_*\tilde{V}_p, W_p = i_*\tilde{W}_p$ for some $\tilde{V}_p, \tilde{W}_p \in T_p\mathbb{S}^2$. Thus

$$(\omega_E)_p(\tilde{V}_p, \tilde{W}_p) = \omega_p(E_p, V_p, W_p) > 0.$$

A similar argument in higher dimensions allows to show that $\mathbb{S}^n$ are orientable manifolds.

Example 8.10. Let us try to understand what happens on $\mathbb{RP}^2$. It is covered by three charts:

$$\varphi_0 : [x_0 : x_1 : x_2] \mapsto (u, v) = (x_1/x_0, x_2/x_0), \quad U_0 = \{x_0 \neq 0\}$$

$$\varphi_1 : [x_0 : x_1 : x_2] \mapsto (a, b) = (x_0/x_1, x_2/x_1), \quad U_1 = \{x_1 \neq 0\}$$

$$\varphi_2 : [x_0 : x_1 : x_2] \mapsto (z, t) = (x_0/x_2, x_1/x_2), \quad U_2 = \{x_2 \neq 0\}.$$

Let us compute the transition maps.

$$\varphi_1 \circ \varphi_0^{-1}(u, v) = \varphi_1[1 : u : v] = (1/u, v/u), \quad \det J(u, v) = -1/u^3,$$

$$\varphi_2 \circ \varphi_0^{-1}(u, v) = \varphi_2[1 : u : v] = (1/v, u/v), \quad \det J(u, v) = 1/v^3,$$

$$\varphi_2 \circ \varphi_1^{-1}(a, b) = \varphi_2[a : 1 : b] = (a/b, 1/b), \quad \det J(a, b) = -1/b^3.$$

This does not yet prove that $\mathbb{RP}^2$ is non-orientable, but it clearly shows the problem with the atlas in question: the expressions for the Jacobian determinants are functions of indeterminate sign.

Let us reason by contradiction. If we ever had a nowhere vanishing $\omega \in \Lambda^2(\mathbb{RP}^2)$, under the chart map $\varphi_1 : U_1 \xrightarrow{\sim} \mathbb{R}^2$, we should have $\omega|_{U_1} = \varphi_1^*(f\, da \wedge db)$ where we can assume $f(a, b) > 0$ for all a, b. Then

$$(\varphi_1 \circ \varphi_0^{-1})^*(f\, da \wedge db)_{(u,v)} = -\frac{1}{u^3} f\left(\frac{1}{u}, \frac{v}{u}\right) du \wedge dv.$$

On the other hand, due to the commutative diagram

$$\begin{tikzcd} & U_0 \cap U_1 \arrow[dr, "\varphi_1"] & \\ \{(u,v)\} \arrow[ur, "\varphi_0^{-1}"] \arrow[rr, "\varphi_1 \circ \varphi_0^{-1}"'] & & \{(a,b)\} \end{tikzcd}$$

it should be equal to $(\varphi_0^{-1})^*(\omega|_{U_0})$ on $u \neq 0$, which is a form $g\, du \wedge dv$ where g is of constant sign. However, from the above we see that $g(u,v) > 0$ for $u < 0$ and $g(u,v) < 0$ for $u > 0$. Thus $\mathbb{RP}^2$ is non-orientable.

The same argument in fact works for any $\mathbb{RP}^{2k}$, which are all non-orientable manifolds. On the other hand, one can also check that *odd-dimensional* $\mathbb{RP}^{2k+1}$ are in fact orientable.

A geometric reason behind the non-orientability of $\mathbb{RP}^2$ is that the maps $\mathbb{S}^2 \to \mathbb{S}^2$ that sends p to $-p$ (antipodal point) reverses, in fact, the orientation of $\mathbb{S}^2$. We then recall that $\mathbb{RP}^2$ is a quotient of $\mathbb{S}^2$ by this map.

More orientable examples: $\mathbb{T}^n$, $\mathbb{CP}^n$, more generally, all complex manifolds, all Lie groups. All symplectic manifolds.

Some non-orientable examples: Möbius strip, Klein bottle.

Note that orientability has nothing to do with embedding into orientable manifolds: we can embed Möbius into $\mathbb{R}^3$ and Klein into $\mathbb{R}^4$ but that will not make them orientable.

8.1 Integrating differential forms

Let us first consider $\mathbb{R}^n$. Let us say that U is an open domain of integration if U is open, bounded and $\partial U = \overline{U} \setminus U$ has (Lebesgue) measure zero. This is entirely a pedantic remark at this point, so if you do not remember measure theory in detail, do not worry.

Such sets have two advantages:

1. Any continuous f defined on an open Ω containing $\overline{U}$ can be Riemann-integrated over U.
2. Any compact subset K of Ω satisfies the following: there exists an open domain of integration U such that $K \subset U \subset \overline{U} \subset \Omega$.

In particular, continuous functions of compact support can be integrated over its support in Riemann or Lebesgue way. Denote $\int_U f dx^1 ... dx^n$ the integral of f over U. When $\operatorname{supp} f = K \subset U$ we will also write $\int_K f dx^1 ... dx^n$.

When $K = [a_1, b_1] \times ... \times [a_n, b_n]$, the famous Fubini theorem guarantees that

$$\int_K f dx^1 ... dx^n = \int_{a_1}^{b_1} dx_1 ... \int_{a_n}^{b_n} f \, dx_n$$

and the integration over x_i can be done in any order.

Let Ω be open in $\mathbb{R}^n$ and U be an open domain of integration whose closure is contained in Ω.

Definition 8.11. For $\omega = f\, dx^1 \wedge ... \wedge dx^n \in \Lambda^n(\Omega)$, its *integral over* U is defined as

$$\int_U \omega := \int_U f\, dx^1 ... dx^n.$$

The following formula perhaps adds some naturality to this definition:

Lemma 8.12. *Let $F : \Theta \to \Omega$ be a diffeomorphism of two opens in $\mathbb{R}^n$, and $V, \overline{V} \subset \Theta$ be an open, connected domain of integration such that $F(V) = U$. Then for $\omega \in \Lambda^n(\Omega)$*

$$\int_V F^*\omega = \pm \int_U \omega,$$

with the sign being negative iff F is orientation-reversing.

Proof. Writing for convenience x^i for coordinates in Ω and y^i for coordinates in Θ, we note that the transformation formula $F^*(f\, dx^1 \wedge ... \wedge dx^n) = \det J(F) \cdot F^*(f)\, dy^1 \wedge ... \wedge dy^n$ looks almost like the change of variable formula for the multiple integral, except that $\det J(F) = \pm |\det J(F)|$ depending on the properties of F. $\square$

Since integrals compute, among other things, volumes of subsets, due to Lemma 8.12, differential forms (and their integrals) are often called oriented volumes.

Let us now extend the integration of forms to manifolds. We shall only consider $\omega \in \Lambda^n(M)$ such that $\operatorname{supp}\omega = \overline{\{p \,|\, \omega_p \neq 0\}}$ is compact in M. Assume, first, that $K = \operatorname{supp}\omega$ is entirely contained in a single chart $\varphi : U \xrightarrow{\sim} \Omega$. Then we define

$$\int_U \omega := \int_{\varphi(K)} (\varphi^{-1})^*\omega.$$

The change of variables formula of Lemma 8.12 then yields:

Lemma 8.13. *The integral $\int_U \omega$ is independent of the choice of the chart map φ, in the following sense: for another chart map $\psi : U \xrightarrow{\sim} \Theta$ that is orientation-compatible with φ, we have*

$$\int_{\varphi(K)} (\varphi^{-1})^*\omega = \int_{\psi(K)} (\psi^{-1})^*\omega. \ \square$$

The orientation remark becomes important if we want to extend it to the whole manifold. To avoid contradictions, integration is usually only defined for orientable M.

We assume that M is oriented, with the choice of orientation given by a nowhere vanishing form that we denote Vol_M. Let (U_i, φ_i) be an orientation-compatible atlas such that $\mathrm{Vol}_M|_{U_i} = g\, d\varphi_i^1 \wedge ... d\varphi_i^n$ with $g > 0$.

Let p_i be a partition of unity subordinate to the cover U_i. Again, this is a bunch of positive smooth functions with $\operatorname{supp} p_i \subset U_i$ and $\sum p_i = 1$, with the sum taken over finite amount of i at each point. Note that for any compact $K \subset M$, the set $\operatorname{supp} p_i \cap K$ is an intersection of a closed and a compact, hence is a compact set (tiny topology exercise) contained in U_i.

Definition 8.14. For $\omega \in \Lambda^n(M)$ with compact support K, we define

$$\int_M \omega = \sum_{i\in I} \int_{U_i} p_i \cdot \omega = \sum_{i \in I} \int_{\varphi_i(\operatorname{supp} p_i \cap K)} (\varphi_i^{-1})^*(p_i \cdot \omega|_{U_i}).$$

Here, I denotes any finite set of indices $i_1, ..., i_k$ such that $K \subset U_{i_1} \cup ... \cup U_{i_k}$.

Obviously, one has to check that

1. this sum is well-defined and does not depend on the finite cover of K by U_i,
2. any other choice of an atlas and a partition of unity gives the same result.

I will not go into detail but these claims are indeed true.

Of course, in practice such a definition is useless for computation, even though we can show that the integral satisfies natural properties like linearity. Still, it allows to prove the following:

Proposition 8.15. *(integrals via parametrisations) Let (M, Vol_M) be an oriented n-manifold and ω a compactly supported form on M. Assume that $D_1, ..., D_k$ are open domains of integration in $\mathbb{R}^n$ and we are given smooth maps $F_i : \overline{D_i} \to M$ (meaning that F_i can be extended to a smooth map from some open containing $\overline{D_i}$ such that*

1. *$W_i = F_i(D_i)$ is open in M and $F_i : D_i \to W_i$ is an orientation-preserving diffeomoprihsm, meaning that $F_i^*(\mathsf{Vol}_M)$ is an n-form on D_i equal to $f\, dx^1 \wedge ... \wedge dx^n$ with $f > 0$,*
2. *$W_i \cap W_j = \emptyset$ for $i \neq j$,*
3. *$\operatorname{supp} \omega \subset \overline{W_1} \cup ... \cup \overline{W_k}$.*

Then $\int_M \omega = \sum_{i=1}^k \int_{D_i} F_i^* \omega$.

It is of course possible to extend this theorem to orientation-reversing F_i, by putting minus signs where needed.

Example 8.16. The map $F : [0, 1] \to \mathbb{S}^1$ that sends α to $\cos\alpha, \sin\alpha$ provides a parametrisation of $\mathbb{S}^1$ in the sense of the previous proposition. We can thus write that $\int_{\mathbb{S}^1} \omega = \int_0^1 F^*\omega$, something that you have already used many times in your life.

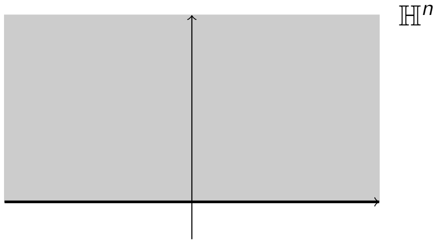
Similarly, we can use the map $F : [0, \pi] \times [0, 2\pi] \to \mathbb{S}^2$, $F(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ to pull back the volume form ω_E constructed earlier in Example 8.9. The result turns out to be $F^*\omega_E = \sin\theta \, d\theta \wedge d\varphi$ and so we get the standard computation

$$\int_{\mathbb{S}^2} \omega = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi.$$

8.2 Manifolds with boundary and Stokes' theorem

There exists an interesting relation between integration and de Rham differential, but to formulate it, it would be useful to generalise our context somewhat, and introduce the notion of manifolds with boundary.

Smooth manifolds are modelled on the opens in $\mathbb{R}^n$ that we consider to be without boundary. The canonical example of a set with boundary is $\mathbb{H}^n := \{(x^1, ..., x^n) \,|\, x^n \geq 0\}$:



Its boundary $\partial\mathbb{H}^n$ is the set $\{(x^1, ..., x^n) \,|\, x^n = 0\}$. The complement of $\partial\mathbb{H}^n$ is the interior $\mathsf{Int}\,\mathbb{H}^n = \mathbb{H}^n \setminus \partial\mathbb{H}^n = \{x^n > 0\}$.

Let $U \in \mathsf{Op}\,\mathbb{H}^n$. We would like to say what it means for $f : U \to \mathbb{R}$ be smooth. First, if U is also open in $\mathbb{R}^n$, which is equivalent to saying $U \cap \partial\mathbb{H}^n = \emptyset$, then we can simply state that f is smooth in the ordinary sense.

If $U \cap \partial\mathbb{H}^n \neq \emptyset$, we need to handle the boundary points. As suggested by our previous constructions, we say that $f : U \to \mathbb{R}$ is smooth if there exists $V \in \mathsf{Op}\,\mathbb{R}^n$, $U \subset V$, and $g \in C^\infty(V)$ such that $g|_U = f$. It turns out that one can rephrase this condition: f should be continuous on U, smooth on $\mathsf{Int}\,\mathbb{H}^n \cap U$, and all the partial derivatives of $f|_{\mathsf{Int}\,\mathbb{H}^n \cap U}$ should admit a continuous extension to U.

We denote $C^\infty_{\mathbb{H}^n}(U)$ to be the set of all smooth functions on $U \in \mathsf{op}\mathbb{H}^n$ in the above sense. One checks that given $U \in \mathsf{Op}\,\mathbb{H}^n$, the assignment $U \mapsto C^\infty_{\mathbb{H}^n}(U)$ gives a $\mathbb{R}$-space structure on $\mathbb{H}^n$.

Definition 8.17. A smooth *manifold with boundary* of dimension n is a (paracompact, Hausdorff) topological space M together with a sheaf of $\mathbb{R}$-algebras denoted C^∞_M such that there exists a cover $M = \cup U_i$ and $\mathbb{R}$-space isomorphisms φ_i between $(U_i, C^\infty_M|_{U_i})$ and either $(\mathbb{H}^n, C^\infty_{\mathbb{H}^n})$ or $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$.

It is possible to replace $\mathbb{H}^n$ by its opens but I will not go into such detail.

For a manifold with boundary M, one says that p is the interior point if its image under some chart map φ belongs to $\mathsf{Int}\,\mathbb{H}^n$, of if p belongs to a chart isomorphic to $\mathbb{R}^n$. It is said to be a boundary point if $\varphi(p) \in \partial\mathbb{H}^n$. One can show that this definition does not depend on the choice of φ, and that an interior point cannot be a boundary point and vice versa.

We denote $\mathsf{Int}\,M$ the set of interior points and ∂M the set of boundary points of M. As we remarked, $\mathsf{Int}\,M \cap \partial M = \emptyset$.

Example 8.18. The canonical example of a manifold with boundary is of course the closed unit ball $\overline{\mathbb{D}^n} \subset \mathbb{R}^n$. One has to do some straightening here to flatten its boundary $\partial\overline{\mathbb{D}^n}$, which happens to be $\mathbb{S}^{n-1}$.

Proposition 8.19. *Let M be a manifold with boundary. Then*

1. $\mathsf{Int}\, M$ *is a smooth manifold without boundary of dimension n,*
2. ∂M *is a smooth manifold without boundary of dimension $n-1$,*
3. *The inclusion $i : \partial M \hookrightarrow M$ is smooth: it is an $\mathbb{R}$-space map, or, equivalently, it takes any $f \in C^\infty_M(M)$ to $i^*f \in C^\infty_{\partial M}(\partial M)$.*

The way ∂M gets charts is similar to slice charts for submanifolds: we find a chart (U, φ) that intersects the boundary and consider $(U \cap \partial M, \varphi|_{U\cap\partial M})$. With our definition of smooth functions on $\mathbb{H}^n$, it is clear that their restriction to $\partial\mathbb{H}^n$ is also smooth. Sheaves allow to globalise to statements about M and ∂M.

Most notions can be defined verbatim in the context of manifolds with boundary. It is worth remarking the following subtlety: for $p \in \partial\mathbb{H}^n$, the tangent space $T_p\mathbb{H}^n$, defined again as p-derivations of $C^\infty(\mathbb{H}^n)$, will be spanned by $\partial_1|_p, ..., \partial_n|_p$ (we can differentiate $f \in C^\infty(\mathbb{H}^n)$ in the n-th direction even at the boundary).

On the other hand, $T_p\partial\mathbb{H}^n = \mathsf{Span}(\partial_1|_p, ..., \partial_{n-1}|_p)$. Similarly, differential forms on $\mathbb{H}^n$ are wedge products of $dx^1, ..., dx^n$ times a smooth function on $\mathbb{H}^n$, whereas on $\partial\mathbb{H}^n$, we will only have $dx^1, ..., dx^{n-1}$. The pull-back of forms along $i : \partial\mathbb{H}^n \hookrightarrow \mathbb{H}^n$ acts as $i^*(dx^n) = 0$ and preserves the remaining dx^j.

Other than this remark we can repeat everything that was said about vector fields, tensor fields and differential forms in the context of manifolds with boundary.

An orientation of a manifold with boundary M consists, again, in specifying a nowhere vanishing top degree form Vol_M. It then turns out that if such a form exists, then ∂M is also orientable. There are various conventions on what orientation of ∂M one should take.

A convention that is employed to orient $\partial\mathbb{H}^n$ as a boundary of $\mathbb{H}^n$ is as follows: $\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^{n-1}}\right)$ is positively oriented on $\partial\mathbb{H}^n$ iff $\left(-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^{n-1}}\right)$ is positively oriented for the standard form $dx^1 \wedge ... \wedge dx^n$ on $\mathbb{H}^n$.

Thus we choose $(-1)^n dx^1 \wedge ... \wedge dx^{n-1}$ as the volume form on $\partial\mathbb{H}^n$ for what is called the induced orientation of the boundary. This can be done similarly for $M, \partial M$ by passing to charts: if $\varphi : U \xrightarrow{\sim} \mathbb{H}^n$ is a chart in which the volume form Vol_M looks like $f\, d\varphi^1 \wedge ... \wedge d\varphi^n$, then we require that $\mathsf{Vol}_{\partial M}$ looks like $(-1)^n g\, d\varphi^1 \wedge ... \wedge d\varphi^{n-1}$, where $g > 0$ on $U \cap \partial M$.

The idea here is that we orient the boundary by adding a vector (field) pointing in the outward direction as our first basis element. This corresponds to how we oriented $\mathbb{S}^1$ and $\mathbb{S}^2$ for example.

These remarks are sufficient to state the Stokes theorem.

Stokes theorem

Theorem 8.20. *Let M be an n-manifold with boundary ∂M and ω be a compactly supported $n-1$ form on M. Then, denoting $i : \partial M \to M$ the smooth inclusion map, we have*

$$\int_M d\omega = \int_{\partial M} i^*\omega.$$

Corollary 8.21. *Let M be a compact manifold without boundary, then for any $n-1$ form ω, we have $\int_M d\omega = 0$. In other words, integration over M induces a linear map*

$$\int_M : H^n_{dR}(M) \to \mathbb{R}$$

since $\mathsf{Vol}_1 = \mathsf{Vol}_2 + d\omega$ *means that* $\int_M \mathsf{Vol}_1 = \int_M \mathsf{Vol}_2$.

Even this corollary is quite nontrivial: changing a form by a differential of another form is a significant perturbation, but the integral does not witness it!

The proof of Stokes theorem even in this generality is surprisingly not complicated. A good write-up is [4, Theorem 16.11], and let us reproduce the computational part of their argument.

By doing the magic of charts and partitions of unity, everything gets reduced to the case of a single chart. It can be of the form $\varphi : U \xrightarrow{\sim} \mathbb{H}^n$ or $\psi : V \xrightarrow{\sim} \mathbb{R}^n$.

The case of φ is treated as follows. Any compactly supported $n-1$ form ω can be written as

$$\omega = \sum_i f_i \, dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

where hats mean omission, and f_i are supported in the interior of $[-R, R]^{n-1} \times [0, R]$ for some $R > 0$ (the $[0, R]$-interval corresponds to the coordinate x^n). One then computes as usual, that

$$d\omega = \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x^i} \, dx^1 \wedge ... \wedge dx^i \wedge ... \wedge dx^n.$$

The integral over $A = [-R, R]^{n-1} \times [0, R]$ can be written as a repeated 1−dimensional integral, so

$$\int_A d\omega = \sum_i (-1)^{i-1} \int_0^R \int_{-R}^R ... \int_{-R}^R \frac{\partial f_i}{\partial x^i} dx^1 ... dx^n.$$

Thanks to Fubini theorem, we can compute in any order we want, so why not integrate $\frac{\partial f_i}{\partial x^i}$ over x^i? Thanks to the fundamental theorem of integration the result will be equal, for $i = 1, ..., n-1$, to $f_i(x^1, ..., R, ..., x^n) - f_i(x^1, ..., -R, ..., x^n)$.

Since we choose R large enough so that $\mathsf{supp}\, f_i$ is contained in the interior of A, both terms in this difference vanish. Similarly, integrating in x^n will give the difference $f_n(x^1, ..., x^{n-1}, R) - f_i(x^1, ..., x^{n-1}, 0)$. Only one term here is nonzero, and that is $-f_n(x^1, ..., x^{n-1}, 0)$. Conclusion:

$$\begin{aligned}\int_A d\omega &= (-1)^n \int_{-R}^R ... \int_{-R}^R f_n(x^1, ..., x^{n-1}, 0) dx^1 ... dx^{n-1} \\ &= \int_{A \cap \partial \mathbb{H}^n} f_n(x^1, ..., x^{n-1}, 0)(-1)^n dx^1 \wedge ... \wedge dx^{n-1}\end{aligned}$$

If we were to use $x^1, \ldots, x^{n-1}$ as coordinates for $\partial\mathbb{H}^n$, then $i^*\omega = f_n(-, \ldots, -, 0)\, dx^1 \wedge \ldots \wedge dx^{n-1}$ so this is almost the expression we have in the integral above. The extra sign is exactly from the orientation discussion we had before, so in terms of how we defined the integration of forms, we indeed have $\int_{\mathbb{H}^n} d\omega = \int_{\partial\mathbb{H}^n} i^*\omega$. Funnily enough, Stokes theorem simply reduces to integrating partial derivatives.

The case of the chart $\psi : V \xrightarrow{\sim} \mathbb{R}^n$ requires similar calculus, with everything being zero since there is no boundary.

Many famous "physics formulas" are in fact corollaries of the Stokes theorem, when suitably interpreted.

Example 8.22. In $\mathbb{R}^2$, we recall that for $\eta = P\, dx + Q\, dy$, we have $d\eta = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$. We can consider an open subset Ω with the property that $\overline{\Omega}$ is a smooth manifold with boundary (for example, take $\Omega = \mathbb{D}^2$). Then, by pulling back η and $d\eta$ to $\overline{\Omega}$

$$\int_{\partial\overline{\Omega}} P\, dx + Q\, dy = \int_{\overline{\Omega}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy,$$

where we suppressed all the pullbacks from the notation. This is Green's theorem. The orientation on $\overline{\Omega}$ is the one induced from $\mathbb{R}^2$, and the induced orientation on the boundary is "counter-clockwise".

I invite you to read up the rest of Green-like theorems as our version of Stokes theorem. It must be noted that one can push to even greater generality, by working with manifolds with corners.

This concludes our course. What a wild ride! I hope you appreciated at least some of it.

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