1. Point-set Topology

Definition 1 (Connectedness)**.** A topological space X is *connected* if for all non-empty $U, V \in \mathsf{Op}(X)$,

$$U \cap V = \emptyset \qquad \Rightarrow \qquad U \cup V \neq X.$$

The maximal connected subsets (ordered by inclusion) of a non-empty topological space are called its *connected components*.

X is said to be *path-connected* if for all $x, y \in X$, there exists a continuous map $\gamma : I \to X$, where I is the unit interval, such that $\gamma(0) = x$ and $\gamma(1) = y$.

(1) Consider X a topological space and Y a set equipped with the discrete topology, that is $\mathsf{Op}(Y) = \mathcal{P}(Y)$. Characterise the continuous maps $f : X \to Y$.

(2) Show that a space X is connected if it is path connected. For an open set $U \subseteq \mathbb{R}^n$, show that the converse holds.

(3) Consider the following subsets of $\mathbb{R}^2$:

$$P_+ := (\mathbb{R} \times \{1\}) \cup \left(\bigcup_{x \in \mathbb{Q}} \{x\} \times I \right),$$

$$P_+ := (\mathbb{R} \times \{-1\}) \cup \left(\bigcup_{x \in \mathbb{Q}} \{x + \sqrt{2}\} \times I \right).$$

Consider $P := P_+ \cup P_-$ equipped with the induced topology. Show that P is connected, but not path-connected.

(4) Here we look at connectedness of two well-known topological groups.
  (a) Consider a finite set of points $Q \subseteq \mathbb{C}$. Show that its complement is connected. Does the same hold if Q is countable?
  (b) Show that $\mathsf{GL}_n(\mathbb{C}) \subseteq \mathbb{C}^{n^2}$ is connected by considering the polynomials
  $$P_{A,B}(z) = \det(zA + (1 - z)B)$$
  for $A, B \in \mathsf{GL}_n(\mathbb{C})$.
  (c) Show that $\mathsf{GL}_n(\mathbb{R})$ is not connected. What are its connected components?

Definition 2 (Product topology)**.** Let $(X, \mathsf{Op}(X))$ and $(Y, \mathsf{Op}(Y))$ be topological spaces. The cartesian product $X \times Y$ of the underlying sets is equipped with *projection maps* p_X and p_y:

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$$

We want to equip this set with a topology. If we only take the products $p_X^{-1}(U) \times p_Y^{-1}(V)$ for $U \in \mathsf{Op}(X)$ and $V \in \mathsf{Op}(Y)$, we will get only the "rectangles" in $X \times Y$, and as we know from $\mathbb{R}^n$, a union of two rectangles is not necessarily a rectangle.

The *product topology* is *generated* by the sets

$$p_X^{-1}(U), \quad U \in \mathsf{Op}(X) \qquad \text{and} \qquad p_Y^{-1}(V), \quad V \in \mathsf{Op}(Y).$$

This is the coarsest topology such that p_X and p_Y are continuous maps. Note that the definition may be generalised to an n-fold or even infinite product.

(5) Let X, Y, Z be topological spaces and consider the product space $X \times Y$.
  (a) Show that p_X and p_Y are open maps.

  (b) Let $f_X : Z \to X$ and $f_Y : Z \to Y$ be continuous maps. Show that there exists a unique continuous map $f : Z \to X \times Y$ such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$.

  (c) Deduce that a map $f : Z \to X \times Y$ is continuous if, and only if, $p_X \circ f$ and $p_Y \circ f$ are continuous.

(6) Show that if $X \times Y$ is connected and non-empty, then both X and Y are connected.

(7) Let X and Y be topological spaces and $f : X \to Y$ a function on the underlying sets. Show that if Y is Hausdorff, the following equivalence holds:

$$f \text{ is continuous} \qquad \Longleftrightarrow \qquad \Gamma(f) \text{ is closed in } X \times Y.$$

(8) Let X, Y be topological spaces. We consider subsets $A \subseteq X$ and $B \subseteq Y$ equipped with the induced topology. The product $A \times B$ may be endowed with:
  i) The induced topology from $X \times Y$.
  ii) The product topology on $A \times B$.

Show that these topologies coincide.

(9) The (2-)Torus $\mathbb{T}^2$ can be defined via the quotient of $\mathbb{R}^2$ by the following equivalence relation:

$$(x, y) \sim (x', y') \Longleftrightarrow (x - x', y - y') \in \mathbb{Z}^2.$$

$\mathbb{T}^2$ can also be seen as a quotient of $[0, 1]^2$ via:

$$(x, 0) \sim (x, 1), (0, y) \sim (1, y) \text{ and } (x, y) \sim (x, y).$$

There are maps $i : [0, 1]^2 \hookrightarrow \mathbb{R}^2$ (inclusion) and $r : \mathbb{R}^2 \to [0, 1]^2$, $r(x, y) = (x - [x], y - [y])$ (fractional parts). These functions respect the equivalence relations, thereby inducing bijections

$$\bar{i} : \mathbb{T} \cong \mathbb{R}^2/\mathbb{Z}^2 : \bar{r}.$$

We can use quotient topology and define topology on $\mathbb{R}^2/\mathbb{Z}^2$ taking the standard topology on $\mathbb{R}^2$. We can do the same for $\mathbb{T}$ using the induced topology on $[0, 1]^2$. Prove that for this choice of topologies, the maps $\bar{i}$ and $\bar{r}$ are continuous (hint: Lemma 1.12 from the course).

(10) Recall from the course that the real projective space $\mathbb{RP}^n$ is defined as the set of all one-dimensional subspaces of $\mathbb{R}^{n+1}$:

$$\mathbb{RP}^n := \{\mathcal{L} \subset \mathbb{R}^{n+1} | \mathcal{L} \text{ is linear and } \dim \mathcal{L} = 1\}.$$

Alternatively we can understand $\mathcal{L} \in \mathbb{RP}^n$ as lines passing through the origin. There are different ways to parametrise these lines.

Take $(x_0, ...x_n) \in \mathcal{L} \setminus 0 \subset \mathbb{R}^n$ (0 denotes the zero subspace). Then every other point of $\mathcal{L}$ can be obtained as $(\lambda x_0, ..., \lambda x_n)$ where we vary $\lambda \in \mathbb{R}$. This motivates to consider $\mathbb{R}^{n+1} \setminus 0$ and put $\sim$ by declaring $(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n)$ for all $\lambda \in \mathbb{R}^*$.

We can then consider the map $q : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ that sends $(x_0, ..., x_n)$ to the line $\mathcal{L} = \{(\lambda x_0, ..., \lambda x_n) | \lambda \in \mathbb{R})$. This induces $\bar{q} : (\mathbb{R}^{n+1} \setminus 0)/_\sim \to \mathbb{RP}^n$. It admits a bijective inverse $g : \mathbb{RP}^n \to (\mathbb{R}^{n+1} \setminus 0)/_\sim$ that sends $\mathcal{L}$ to $[(y_0, ..., y_n)]$ where $(y_0, ..., y_n)$ is a nonzero vector of $\mathcal{L}$.

(a) Using the bijection, endow $\mathbb{RP}^n$ with a topology such that $U \subset \mathbb{RP}^n$ is open if, and only if, $\bar{q}^{-1}(U) = g(U)$ is open in $\mathbb{R}^{n+1}/_\sim$.

(b) Consider the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^n$:

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | \, \|x\| = 1\}.$$

We consider the *antipodal equivalence relation* $\sim$ on $\mathbb{S}^n$ defined by:

$$x \sim y \qquad \iff \qquad \begin{cases} x = -y \\ x = y. \end{cases}$$

Show that $\mathbb{S}^n / \sim$ is homeomorphic to $\mathbb{RP}^n$.

Definition 3. Let X be a topological space. An *open cover* of X is a family $C = \{U_i\}_{i \in I}$ of open sets such that

$$\bigcup_{i \in I} U_i \supseteq X.$$

A *subcover* of C is a subfamily $C' = \{U_j\}_{j \in J}$, where $J \subseteq I$, such that C' is an open cover of X. An open cover of X is finite if the indexing set I is finite.

A space X is said to be *compact* if for every open cover $C = \{U_i\}_{i \in I}$ of X, there exists a finite subcover C' of X. In other words, there exist $i_1, \ldots, i_n \in I$ such that

$$X = U_{i_1} \cup \cdots \cup U_{i_n}.$$

(11) Show that a subset $U \subseteq \mathbb{R}^n$ is compact if, and only if, it is closed and bounded.

(12) Show that $\mathbb{RP}^n$ is compact.

(13) Show that if X is compact and $F \subseteq X$ is closed, then F is compact.

(14) Show that if Y is compact, then the projection onto X

$$p_X : X \times Y \longrightarrow X$$

is a closed map.

(15) Show that $X \times Y$ is non-empty and compact if and only if both X and Y are compact.

(16) Show that if $(x_n)_{n \in \mathbb{N}}$ is a convergent real sequence, then the set

$$\{x_1, x_2, \ldots, x_n, \ldots\}$$

is a compact subset of $\mathbb{R}$.

2. Sheaves and manifolds

(1) Recall from the course that

> a *topological n-manifold* is a Hausdorff topological space M such that for any point $x \in M$ there exists $U \in \mathsf{Op}(M)$ containing x and a homeomorphism between U and an open subset Ω of $\mathbb{R}^n$.

Show that it is equivalent to require that

- for any point $x \in M$ there exists $U \in \mathsf{Op}(M)$ containing x, $\varepsilon > 0$ and a homeomorphism between U and an open ball $B(0, \varepsilon)$ of $\mathbb{R}^n$.
- for any point $x \in M$ there exists $U \in \mathsf{Op}(M)$ containing x, and a homeomorphism between U and $\mathbb{R}^n$.

(2) Show that a topological manifold is connected if, and only if, it is path connected.

(3) Let T and X be topological spaces.

(a) For $U \in \mathsf{Op}(X)$, let $\mathcal{P}(U)$ be the set of constant functions $U \to T$. Show that this is a pre-sheaf. Why is it not a sheaf?

(b) Recall that a function is *locally constant* if it is constant on a neighbourhood of every point. Show that the assignment of $U \in \mathsf{Op}(X)$ to the set $\mathcal{S}(U)$ of locally constant functions is a sheaf.

(c) For what topology on T do we have $C(X, T) = \mathcal{F}(X, T)$?

(d) Give an example of a space X for which $\mathcal{P}$ and $\mathcal{S}$ coincide.

(4) Sheafification, pushing and pulling...

(a) Let $\mathcal{P}$ be a presheaf of T-valued functions on X a topological space. The *sheafification* $\mathcal{P}_s$ of $\mathcal{P}$ assigns $U \in \mathsf{Op}(X)$ to the set

$$\mathcal{P}_s(U) = \{f : U \to T | \forall x \in U, \exists U_x \in \mathsf{Op}(U), \text{ s.t. } x \in U_x \text{ and } f|_{U_x} \in \mathcal{P}(U_x)\}.$$

Show that $\mathcal{P}_s$ is a sheaf.

(b) Let $\mathcal{S}$ a sheaf of T-valued functions on a space X and let $\varphi : X \to Y$ a continuous map. We define the *push-forward* of $\mathcal{S}$ along φ, denoted by $\varphi_*\mathcal{S}$:

$$\varphi_*\mathcal{S}(V) = \mathcal{S}(\varphi^{-1}(V)).$$

Show that $\varphi_*\mathcal{S}(V)$ is a sheaf.

(c) Can we define a "pull-back" for sheaves?

(5) Let $(X, \mathcal{A}), (Y, \mathcal{B}), (Z, \mathcal{C})$ be $\mathbb{R}$-spaces and consider morphisms

$$\varphi : (X, \mathcal{A}) \to (Y, \mathcal{B}), \quad \text{and} \quad \varphi' : (Y, \mathcal{B}) \to (Z, \mathcal{C})$$

(a) Show that the composition $\varphi' \circ \varphi$ is again a morphism of $\mathbb{R}$-spaces.

(b) Show that the following are equivalent:

- φ is an isomorphism of $\mathbb{R}$-spaces (as defined in the course).
- φ admits an inverse morphism, that is there exists a morphism

$$\psi : (Y, \mathcal{B}) \to (X, \mathcal{A})$$

such that

$$\psi \circ \varphi = id_{(X,\mathcal{A})} \quad \text{and} \quad \varphi \circ \psi = id_{(Y,\mathcal{B})}.$$

(6) Let $\Omega \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$ be open sets considered as $\mathbb{R}$-spaces with their sheaves of C^k $\mathbb{R}$-valued functions.

Show that a $\mathbb{R}$-space morphism

$$\varphi : (\Omega, C^k_\Omega) \to (\Theta, C^k_\Theta)$$

is C^k in the ordinary sense, *i.e.* has continuous derivatives up to rank k.

(7) Recall from the course the local property required of a manifold $(M, \mathcal{A})$:

> For each $x \in M$ there exists $U \in \mathsf{Op}(M)$, $x \in U$, and a $\mathbb{R}$-space isomorphism $(U, \mathcal{A}|_U) \xrightarrow{\sim} (\Omega, C^k_\Omega)$ where $\Omega \in \mathsf{Op}(\mathbb{R}^n)$.

Verify that it is equivalent to require that for each $x \in M$ there exists $U \in \mathsf{Op}(M)$, $x \in U \ldots$

(a) and a $\mathbb{R}$-space isomorphism $(U, \mathcal{A}|_U) \xrightarrow{\sim} (B(0, \varepsilon), C^k_B)$.

(b) and a $\mathbb{R}$-space isomorphism $(U, \mathcal{A}|_U) \xrightarrow{\sim} (\mathbb{R}^n, C^k)$.

(8) Show that $\mathbb{CP}^n$ is a smooth manifold (Example 2.23 could provide inspiration. . .) Furthermore, show that there is a diffeomorphism $\mathbb{CP}^1 \cong \mathbb{S}^2$.

(9) The n-dimensional torus $\mathbb{T}^n$ is defined as the quotient of $\mathbb{R}^n$ by the equivalence relation

$$(x_1, \ldots, x_n) \sim (x'_1, \ldots, x'_n) \iff (x_1 - x'_1, \ldots, x_n - x'_n) \in \mathbb{Z}^n.$$

Show that $\mathbb{T}^n$ is a smooth manifold.

3. More on manifolds and smooth maps

(1) Consider
$$\mathsf{SL}_2(\mathbb{R}) = \{M \in \mathsf{GL}_2(\mathbb{R}) \quad | \quad \det M = 1\}.$$
Show (without using theorems of section 4) that $\mathsf{SL}_2(\mathbb{R})$ has a structure of a smooth 3-manifold for which the inclusion $i : \mathsf{SL}_2(\mathbb{R}) \to \mathsf{GL}_2(\mathbb{R})$ is a smooth map.

(2) For the smooth structure found above, prove that $i_* : T_{I_2}\mathsf{SL}_2(\mathbb{R}) \to T_{I_2}\mathsf{GL}_2(\mathbb{R})$ is injective and compute its image. Here I_2 denotes the identity matrix.

(3) Let M, N be two smooth manifolds. Show that there exists a smooth structure on $M \times N$ such that
  (a) The projection maps $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N$ are smooth.
  (b) Given two smooth maps $F : K \to M$ and $G : K \to N$ from a third smooth manifold K, there exists unique $H : K \to M \times N$ that is smooth and satisfies $\pi_1 \circ H = F, \pi_2 \circ H = G$.

(4) Let M, N be two smooth manifolds and $M \times N$ carry a smooth structure of previous exercise. Show that for $p \in M$, the map $N \to M \times N$ defined as $x \mapsto (p, x)$ is smooth.

(5) Consider $\mathbb{R}^3$ and $p \neq (0,0,0)$. Let $V = A\frac{\partial}{\partial x}\big|_p + B\frac{\partial}{\partial y}\big|_p + C\frac{\partial}{\partial z}\big|_p$ be a tangent vector in $T_p\mathbb{R}^3$. Compute F_*V where
  (a) $F = u : v \mapsto v/\|v\|$, $\mathbb{R}^3 \setminus 0 \to \mathbb{R}^3$.
  (b) $F = E \circ \pi$ where $\pi(x,y,z) = (x,y)$ and
$$E(x,y) = ((2+\cos 2\pi x)\cos 2\pi y, (2+\cos 2\pi x)\sin 2\pi y, \sin 2\pi x)$$
  read as a map $\mathbb{R}^2 \to \mathbb{R}^3$.

(6) Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ and denote $N = (0,0,1)$ the northern pole.
  (a) Denote $\sigma : \mathbb{S}^2 \setminus N \to \mathbb{R}^2$ the stereographic projection onto the equatorial plane. Prove that $\sigma : (\mathbb{S}^2 \setminus N, C^\infty_{\mathbb{S}^2\setminus N}) \to (\mathbb{R}^2, C^\infty_{\mathbb{R}^2})$ is a diffeomorphism.
  (b) Let $p = (1,0,0)$ and $V \in T_p\mathbb{S}^2$ be such that $\sigma_*V = \frac{\partial}{\partial x}\big|_{(1,0)} + \frac{\partial}{\partial y}\big|_{(1,0)}$. Compute $(\varphi_0^+)_*V$ where φ_0^+ is the open chart map considered in lectures.
  (c) Take same p, V and compute μ_*V, where μ is the stereographic projection from the southern pole $S = (0,0,-1)$.

(7) Let M be a compact smooth manifold and $F : M \to \mathbb{R}$ a smooth function. Prove that $F_* : T_pM \to T_{F(p)}\mathbb{R} \cong \mathbb{R}$ is zero map for some $p \in M$.

(8) Recall the definition of Hopf fibration from the lecture notes. Show that it is a smooth map $\mathbb{S}^3 \to \mathbb{S}^2$ and compute its rank.

(9) Let (M, C^∞_M) be a smooth manifold. Prove that the three definitions of the tangent space T_pM are equivalent:
  (a) As the set of p-derivations of $C^\infty(M)$,
  (b) As the set of p-derivations of smooth germs algebra C^∞_p,

(c) As the set of equivalence classes of curves passing through p, with two curves being equivalent if they produce the same derivative at p (computed in some chart).

(10) Let $F : \mathbb{R} \to \mathbb{R}^2$ be the map $t \mapsto (t, \alpha t)$ where $\alpha \in \mathbb{R}$ is irrational. Consider $G = q \circ F$ where $q : \mathbb{R}^2 \to \mathbb{T}$ is the smooth quotient map. Prove that G is an injective immersion. Can G be an embedding?

(11) Consider the smooth map

$$\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4, \quad ((x_0, x_1), (y_0, y_1)) \mapsto (x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1).$$

Show that σ induces a smooth immersion $S : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^3$. Describe the image of S.

(12) Using theorems from the lectures, show that $\mathsf{SL}_n(\mathbb{R})$ is a smooth submanifold of $\mathsf{GL}_n(\mathbb{R})$, compute its dimension and tangent space at the identity matrix I_n. Same question for $\mathsf{O}_n(\mathbb{R})$.

(13) Let $F : M \to N$ be a surjective submersion and $f : N \to \mathbb{R}$ a function. Prove that $f \in C^\infty(N) \iff F^*(f) \in C^\infty(M)$.

4. Vector fields

(1) Consider the inclusion $i : S^n \subset \mathbb{R}^{n+1}$. Let U be an open set of $\mathbb{R}^{n+1}$ containing the image of i and $X \in \mathcal{T}_{\mathbb{R}^{n+1}}(U)$ a vector field. Assume further that for each $p \in \mathbb{S}^n$, X_p is tangent to $\mathbb{S}^n$. In other words, we have $X_p \in \operatorname{Im} i_*(p)$.

Show that in this situation, the assignment $p \mapsto Y_p = i_*(p)^{-1} X_p$ defines a smooth vector field on $\mathbb{S}^n$.

Indication. $Y(f) = Y(i^* u^* f) = X(u^* f|_U)$.

(2) Use the previous exercise to find a nowhere vanishing vector field on $\mathbb{S}^1$.

Indication. Apply the previous observation to $x\partial_y - y\partial_x$.

(3) Recall $\sigma : U = \mathbb{S}^2 \setminus N \to \mathbb{R}^2$, the stereographic projection from the north pole considered in the previous file, and $\mu : V = \mathbb{S}^2 \setminus S \to \mathbb{R}^2$ the stereographic projection from the southern pole.

(a) Let $X \in \mathcal{T}_{\mathbb{S}^2}(U)$ be a vector field on U such that $\sigma_* X$ is the partial derivative along the first coordinate of $\mathbb{R}^2$. Find the formula for $\mu_*(X|_{U\cap V})$.

(b) Same question for X such that $\sigma_* X$ is the partial derivative along the second coordinate of $\mathbb{R}^2$.

(c) Find a vector field on $\mathbb{S}^2$ with exactly one zero.

Indication. Use the computations from PS3.

(4) The quaternion algebra $\mathbb{H}$ is a unital associative $\mathbb{R}$-algebra generated by the elements $1, i, j, k$ satisfying the following relations: 1 is the unit and

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \quad i^2 = j^2 = k^2 = -1.$$

For a quaternion $q = a1 + bi + cj + dk$, we denote $\|q\|^2 := a^2 + b^2 + c^2 + d^2$.

(a) Give a formula for multiplication $q \cdot q'$ of two $q, q' \in \mathbb{H}$.

(b) Show that for each $q \in \mathbb{H} \setminus 0$, one can find inverse q^{-1}.

(c) Consider the isomorphism $B : \mathbb{R}^4 \xrightarrow{\sim} \mathbb{H}$, $(x^0, x^1, x^2, x^3) \mapsto x^0 1 + x^1 i + x^2 j + x^3 k$. Describe the image of $\mathbb{S}^3$ under B.

(d) Using the isomorphism B, we can look at i, j, k as linear maps $\mathbb{R}^4 \to \mathbb{R}^4$. Write their associated matrices.

(e) Consider the assignments

$$X(x) = \sum_m (i \cdot x)^m \frac{\partial}{\partial x^m}, \quad Y(x) = \sum_m (j \cdot x)^m \frac{\partial}{\partial x^m}, \quad Z(x) = \sum_m (k \cdot x)^m \frac{\partial}{\partial x^m},$$

show that these formulas define three smooth vector fields on $\mathbb{R}^4$ linearly independent at each point and tangent to $\mathbb{S}^3$ in the sense described in Problem 1.

(f) Show that there are three nowhere vanishing vector fields on $\mathbb{S}^3$, linearly independent at each point.

Indication. Nothing to see here, perhaps check linear independence of X, Y, Z on $\mathbb{R}^4$. Can also remark that the same method works for $\mathbb{S}^1 \subset \mathbb{C}$.

(5) Let M be a smooth manifold of dimension n and $s_1, \ldots s_n \in \mathcal{T}_M(M)$ be smooth vector fields satisfying the following: for each $p \in M$, $\operatorname{Span}((s_1)_p, \ldots, (s_n)_p) = T_p M$.

(a) Show that any $X \in \mathcal{T}_M(M)$ can be expressed as a sum $X = \sum_i f_i s_i$, where $f_1, \ldots, f_n \in C^\infty_M(M)$.

(b) Conclude that as a $C^\infty(M)$-module, $\mathcal{T}_M(M)$ is isomorphic to $(C^\infty(M))^n$.

Indication. Writing X as such a sum is definitely possible since s_i span tangent spaces everywhere. To see why f_i are smooth, we could proceed locally. On an open chart U, φ one has $\varphi_* s_i|_U = \sum_j s_i^j \partial_j$. The assignment $p \mapsto (s_i^j(\varphi(p)))$ is a smooth map $S : U \to \mathsf{Mat}_n(\mathbb{R})$, that in fact factors through $\mathsf{GL}_n(\mathbb{R})$. Since inversion operation is smooth on GL_n the pointwise inverse $A : p \mapsto S(p)^{-1}$ is also smooth on U. It can be used to transition from $X = \sum X^i \varphi_*^{-1} \partial_i$ to $\sum f_i s_i|_U$ (mapping X^i by A will forcefully give f_i due to the uniqueness of the decomposition in $T_p M$).

(6) Derive the Lie bracket formula for vector fields on $\Omega \subset \mathbb{R}^n$.

(7) Let $X, Y \in \mathcal{T}(M)$ and $f, g \in C^\infty(M)$. Show that

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$$

in $\mathcal{T}(M)$.

Indication. For the love of, please do not use coordinates here, it is not necessary. Maybe compute first $[fX, Z]$ and $[Z, gY]$.

(8) A function $f \in C^\infty(\mathbb{R}^{n+1} \setminus 0)$ is c-homogeneous ($c \in \mathbb{R}$) if for any $\lambda \geq 0$ and $x \in \mathbb{R}^{n+1} \setminus 0$, one has

$$f(\lambda x) = \lambda^c f(x).$$

(a) Show that there exists unique vector field $E \in \mathcal{T}(\mathbb{R}^{n+1} \setminus 0)$ such that $E(f) = cf$ for each c-homogeneous function f.

(b) Show that E is related to the zero vector fields on $\mathbb{S}^n$ and $\mathbb{RP}^n$ by the means of maps $u : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$ and $q : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$.

Indication. It is easy to see, by applying E to coordinate functions, that E has to be the Euler vector field. Then one needs to compute as in `https://math.stackexchange.com/questions/1932168/euler-vector-field-and-homogeneous-function`. The second point is easy since functions on $\mathbb{S}^n$ are 0-homogeneous etc.

(9) Let $F : \mathbb{R}^2 \to \mathbb{RP}^2$ be the map defined as $(x, y) \mapsto [x : y : 1]$ (the inverse of a standard chart). Consider $V = x\partial_y - y\partial_x$, a vector field on $\mathbb{R}^2$. Show that there is a vector field $W \in \mathcal{T}(\mathbb{RP}^2)$ that is F-related to V.

Indication. A really nice solution uses flows: `https://math.stackexchange.com/questions/2544726/finding-a-f-related-field-in-mathbbrp2`. There is no need to understand theory of flows in order to use it!

(10) Let M be a smooth manifold. Show that $TM = \sqcup T_p M$ can be equipped with a smooth structure such that smooth vector fields correspond to smooth maps – sections of $\pi : TM \to M$. Show that such a smooth structure is unique.

(1) Let M be a smooth manifold. In lecture notes, we defined the smooth 1-form, or 1-tensor field df by the rules $df(X) = X(f)$, $df_p(X_p) = X_p(f)$. Show that the operation $f \mapsto df$ satisfies the following:

(a) $d(af + bg) = adf + bdg$ $f, g \in C^\infty(M)$, $a, b \in \mathbb{R}$.

(b) $d(fg) = gdf + fdg$, $f, g \in C^\infty(M)$

(c) Let $g \in C^\infty(M)$ and $U = M \setminus \{p|g(p) = 0\}$. Then on U, $d(f/g) = (gdf - fdg)/g^2$ for $f \in C^\infty(U)$.

(d) If $f \in C^\infty(M)$ with $\mathsf{Im}\, f \subset I \in \mathsf{Op}\,\mathbb{R}$ and $h \in C^\infty(I)$, then $d(h \circ f) = (h' \circ f)df$

(e) $df = 0$ *if and only if* f is constant.

Indication. Many points are obvious, the important part here is to not reduce to coordinates when unnecessary.

(a) Trivial check.
(b) Trivial check using derivation property of vector fields.
(c) It is worth explaining how one defines $1/g$ on U. I might be too asleep but existence of inverses in ambient algebras of functions does not imply that these inverses exist in subalgebras. The proof here is that if $g \neq 0$ on some chart V then $1/g \in C^\infty(V)$ since its pullback is smooth on $\varphi(V)$ (we know this result for smooth functions on opens in $\mathbb{R}^n$). Then we conclude using sheaf.
With this out of the way, the computation is like in (b).
(d) It is better to check pointwise: for each $X_p \in T_pM$, have

$$d(h \circ f)_p(X_p) = X_p(h \circ f) = f_* X_p(h)$$

The vector $Y_{f(p)} = f_* X_p \in T_{f(p)}\mathbb{R}$ acts as $h \mapsto A dh/dt(f(p))$. To compute A, it suffices to plug in $h = \mathsf{id}_\mathbb{R}$. But $f_* X_p(\mathsf{id}_\mathbb{R}) = df_p(X_p)$ by definition.
(e) The nontrivial part here is "only if". I reproduce the proof of Lee. WLOG assume M connected. Let us suppose $df = 0$. Let $p \in M$ and let $\mathcal{C} = \{q \in M | f(q) = f(p)\} = f^{-1}(f(p))$. For any point q in $\mathcal{C}$ take a connected chart U, φ containing q. Then we have that $0 = (\varphi^{-1})^* df = d(f \circ \varphi^{-1})$ on $\Omega = \varphi(U)$. But for any $g \in C^\infty(\Omega)$, we have $dg = \sum \partial_i g dx^i$. So $dg = 0$ means $\partial_i g = 0$ for all i as functions on Ω. This means that g is locally constant, or constant tout court since Ω is connected. Thus $\mathcal{C}$ is an open set and hence it has to coincide with M.

(2) Let $F : M \to \mathbb{R}^m$ be a smooth map. For $p \in M$, express $F_*(p) : T_pM \to T_{F(p)}\mathbb{R}^m$ in terms of $dF^1_p, \ldots, dF^m_p$.

Indication. Comment: F is smooth iff all F^i are. Proof: if we restrict to a chart, then this amounts to saying $F \circ \varphi^{-1}$ smooth iff $F^i \circ \varphi^{-1}$ smooth, but this is a known fact from calculus.

Let $V \in T_pM$. Each dF^i_p acts as $DF^i_p(V) = V(F^i_p)$. Denote $y^i : \mathbb{R}^m \to \mathbb{R}$ the coordinate functions. Note that $F_*(p)(V)(y^i) = V(F^i) = dF^i_p(V)$ and since $F_*(p)(V) =$

$\sum F_*(p)(V)(y^i)\partial_{y^i}|_{F(p)}$, we have

$$F_*(p)(V) = \sum dF^i_p(V)\partial_{y^i}|_{F(p)} = \left(\sum dF^i_p \otimes \partial_{y^i}|_{F(p)}\right)(V).$$

The latter notation is for internal use. But in principle one can explain that given two vector spaces A, B, a set of m linear forms $f^1, \dots, f^m$ on A and m vectors $v_1, \dots, v_m$ in B, the notation $\sum f^i \otimes v_i$ denotes the linear operator $w \in A \mapsto \sum f^i(w)v_i$. It is a $(1,1)$-tensor, something that is slightly outside our scope in this course.

(3) Consider $\mathbb{R}^3$ and consider the Riemannian metric $g \in \otimes^2\mathcal{T}^*(\mathbb{R}^3)$, $g = dx^{\otimes 2} + dy^{\otimes 2} + dz^{\otimes 2}$. As usual, $i : \mathbb{S}^2 \subset \mathbb{R}^3$ denotes the smooth inclusion and $g_r := i^*g$.

(a) Compute $(\varphi^{-1})^*g_r$ where $\varphi : \mathbb{S}^2 \cap \{z > 0\} \to \mathbb{D}^2, (x, y, z) \mapsto (x, y)$ is the vertical projection chart.

Solution. One has $(\varphi^{-1})^*g_r = (\varphi^{-1})^*i^*g = (i \circ \varphi^{-1})^*g$. The map $\eta = i \circ \varphi^{-1} : \mathbb{D}^2 \to \mathbb{R}^3$ acts as $(u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$. Write $\eta = (\eta^1, \eta^2, \eta^3)$. Using that $\eta^*df = d(f \circ \eta)$ (Proposition 6.23.1), we see that $\eta^*(dx) = d\eta^1$, $\eta^*(dy) = d\eta^2$, $\eta^*(dz) = d\eta^3$, and so $\eta^*g = d\eta^1 \otimes d\eta^1 + d\eta^2 \otimes d\eta^2 + d\eta^3 \otimes d\eta^3$. It remains to express it in terms of du and dv. We first find

$$d\eta^1 = du, \quad d\eta^2 = dv, \quad d\eta^3 = \frac{-1}{\eta^3}(udu + vdv),$$

where u and v mean the coordinate functions $(u, v) \mapsto u$, $(u, v) \mapsto v$. Summing all this, we find, that at a point (u, v),

$$\begin{aligned}\eta^*(g)_{(u,v)} &= du \otimes du + dv \otimes dv \\ &\quad + \frac{1}{1 - u^2 - v^2}(u^2 du \otimes du + v^2 dv \otimes dv + uv(du \otimes dv + dv \otimes du)) \\ &= \frac{1 - v^2}{1 - u^2 - v^2}du^{\otimes 2} + \frac{1 - u^2}{1 - u^2 - v^2}dv^{\otimes 2} + \frac{uv}{1 - u^2 - v^2}(du \otimes dv + dv \otimes du).\end{aligned}$$

Strictly speaking, we should write $du^{\otimes 2}_{(u,v)}$ etc, but this is a subtlety that I am willing to look over. The appearance of the "diagonal term" with $du \otimes dv$ corresponds to the fact that this kind of projection does not preserve angles very well. As the computation below shows, the stereographic projection is much nicer, even if one has to struggle to get the results.

(b) Compute $(\sigma^{-1})^*g_r$ where $\sigma : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$, $(x, y, z) \mapsto \frac{1}{1-z}(x, y)$ is the stereographic projection from the northern pole.

Solution. Same sauce different map. The inverse of σ is given in PS3. Repeating the same argument, we see that we need to compute F^*g, where

$$F(u, v) = (F^1(u, v), F^2(u, v), F^3(u, v)) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2}\right)$$

We need to compute

$$dF^1_{(u,v)} = \frac{2du}{1 + u^2 + v^2} - \frac{4u(udu + vdv)}{(1 + u^2 + v^2)^2}$$

$$dF^2_{(u,v)} = \frac{2dv}{1 + u^2 + v^2} - \frac{4v(udu + vdv)}{(1 + u^2 + v^2)^2}$$

$$dF^3_{(u,v)} = \frac{2udu + 2vdv}{1 + u^2 + v^2} + \frac{1 - u^2 - v^2}{(1 + u^2 + v^2)^2}(2udu + 2vdv) = \frac{4}{(1 + u^2 + v^2)^2}(udu + vdv)$$

If you are computation expert, you can then compute head-on $\sum dF^i \otimes dF^i$. I would rather like to see how these differentials interact with ∂_u and ∂_v:

$$dF^1_{(u,v)}(\partial_u) = \frac{2}{1 + u^2 + v^2} - \frac{4u^2}{(1 + u^2 + v^2)^2} = \frac{2 - 2u^2 + 2v^2}{(1 + u^2 + v^2)^2},$$

$$dF^2_{(u,v)}(\partial_u) = -\frac{4uv}{(1 + u^2 + v^2)^2}, \quad dF^3_{(u,v)}(\partial_u) = \frac{4u}{(1 + u^2 + v^2)^2}.$$

Similarly,

$$dF^1_{(u,v)}(\partial_v) = -\frac{4uv}{(1 + u^2 + v^2)^2},$$

$$dF^2_{(u,v)}(\partial_v) = \frac{2 + 2u^2 - 2v^2}{(1 + u^2 + v^2)^2}, \quad dF^3_{(u,v)}(\partial_v) = \frac{4v}{(1 + u^2 + v^2)^2}.$$

We now compute:

$$F^*g_{(u,v)}(\partial_u, \partial_v) = dF^1_{(u,v)}(\partial_u) \cdot dF^1_{(u,v)}(\partial_v) + dF^2_{(u,v)}(\partial_u) \cdot dF^2_{(u,v)}(\partial_v) + dF^3_{(u,v)}(\partial_u) \cdot dF^3_{(u,v)}(\partial_v) \tag{1}$$

$$= \frac{-4uv(2 - 2u^2 + 2v^2) - 4uv(2 + 2u^2 - 2v^2) + 16uv}{(1 + u^2 + v^2)^4} = 0. \tag{2}$$

If you have some intuition about the stereographic projection, then seeing this should not be surprising: it preserves angles! Next,

$$F^*g_{(u,v)}(\partial_u, \partial_u) = \frac{(2 - 2u^2 + 2v^2)^2 + 16u^2v^2 + 16u^2}{(1 + u^2 + v^2)^4} = \frac{4}{(1 + u^2 + v^2)^2}$$

And since u, v play symmetrically,

$$F^*g_{(u,v)}(\partial_v, \partial_v) = \frac{(2 + 2u^2 - 2v^2)^2 + 16u^2v^2 + 16v^2}{(1 + u^2 + v^2)^4} = \frac{4}{(1 + u^2 + v^2)^2}$$

Now, any Riemannian metric h in u, v decomposes as $h = Adu^{\otimes 2} + Bdv^{\otimes 2} + C(du \otimes dv + dv \otimes du)$. Here, $A = h(\partial_u, \partial_u)$, $B = h(\partial_v, \partial_v)$, $C = h(\partial_u, \partial_v)$. For $h = F^*g$, our computations show that $C = 0$, $A(u, v) = B(u, v) = 4/(1 + u^2 + v^2)^2$. Thus

$$F^*g_{(u,v)} = \frac{4}{(1 + u^2 + v^2)^2}(du^{\otimes 2} + dv^{\otimes 2}).$$

(c) Consider $P : \{(\varphi, \theta)\} = \mathbb{R}^2 \to \mathbb{R}^3$, $P(\varphi, \theta) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. Explain why P can be viewed as a smooth map $P : \mathbb{R}^2 \to \mathbb{S}^2$ and compute $P^*(g_r)$.

Solution. P is smooth since it is written using smooth functions, and so in each standard chart $\varphi_i^{\pm}$ we will get a map $\varphi_i^{\pm} \circ P$ written using smooth functions. Another idea is that $u \circ P = P$ with $u : \mathbb{R}^3 \setminus 0 \to \mathbb{S}^2$, so $P^*f = P^*(u^*f)$ is smooth for any smooth $f : \mathbb{S}^3 \to \mathbb{R}$.

Anyway, to compute $P^*(g_r)$ is the same as to compute $P^*(g)$ where I read P again as a map $\mathbb{R}^2 \to \mathbb{R}^3$. Writing $P = (P^1, P^2, P^3)$ we find

$$dP^1 = -\sin\varphi \sin\theta d\varphi + \cos\varphi \cos\theta d\theta, \; dP^2 = \cos\varphi \sin\theta d\varphi + \sin\varphi \cos\theta d\theta,$$

$$dP^3 = -\sin\theta d\theta.$$

Here we abuse the notation slightly, meaning $dP^i_{(\varphi,\theta)}$. It remains to compute $P^*g = dP^1 \otimes dP^1 + dP^2 \otimes dP^2 + dP^3 \otimes dP^3$. One finds:

$$dP^1 \otimes dP^1 = \sin^2\varphi \sin^2\theta d\varphi \otimes d\varphi + \cos^2\varphi \cos^2\theta d\theta \otimes d\theta - \sin\varphi\cos\varphi\sin\theta\cos\theta(d\varphi \otimes d\theta + d\theta \otimes d\varphi)$$

$$dP^2 \otimes dP^2 = \cos^2\varphi \sin^2\theta d\varphi \otimes d\varphi + \sin^2\varphi \cos^2\theta d\theta \otimes d\theta + \sin\varphi\cos\varphi\sin\theta\cos\theta(d\varphi \otimes d\theta + d\theta \otimes d\varphi)$$

$$dP^3 \otimes dP^3 = \sin^2\theta d\theta \otimes d\theta$$

$$\sum dP^i \otimes dP^i = (\sin^2\varphi + \cos^2\varphi)\sin^2\theta d\varphi \otimes d\varphi + (\sin^2\varphi + \cos^2\varphi)\cos^2\theta d\theta \otimes d\theta + \sin^2\theta d\theta \otimes d\theta.$$

We thus get that $P^*g = \sin^2\theta d\varphi \otimes d\varphi + d\theta \otimes d\theta$. This classic formula has the intuition behind it that different θ-angles correspond to different radii of φ-circles.

(d) Show that P restricted to $\varphi \in]0, 2\pi[, \theta \in]0, \pi[$ is an inverse of a coordinate chart.

Indication. We know that (unit radius) spherical coordinates provide a bijection between $\mathbb{S}^2$ and $[0, 2\pi[\times[0, \pi]$: for $(x, y, z) \in \mathbb{S}^2$ there is unique $\theta \in [0, \pi]$ such that $z = \cos\theta$, and then $x^2+y^2 = 1-z^2 = \sin^2\theta$ so we can uniquely write $x = \cos\varphi\sin\theta$, $y = \sin\varphi\sin\theta$. Thus $\theta = \arccos z$ and for $\theta \neq 0, \pi$, we have $\varphi = \arctan 2(y, x)$ where $\arctan 2$ is the 2-argument inverse tangent function. It is smooth, as per `https://en.wikipedia.org/wiki/Atan2` (wiki version needs an argument shift).

We consider $P :]0, 2\pi[, \times]0, \pi[\to \mathbb{S}^2$, whose image is $\mathbb{S}^2$ minus the set $\{\sin\theta, 0, \cos\theta | \theta \in [0, \pi]\}$, we denote it $U = P(]0, 2\pi[, \times]0, \pi[) = \mathbb{S}^2 \cap (\mathbb{R}^3 \setminus \{(x, 0, z\})$, so it is open. By the previous paragraph, you are either convinced that $P|_{]0,2\pi[,\times]0,\pi[}$ is bijective with smooth inverse, or that it is simply bijective (and the inverse smoothness is not clear, especially since we are dealing with $\mathbb{S}^2$ and not $\mathbb{R}^3$). If the latter is the case, note that it is a rank 2 map (easy check using the Jacobian) so by Corollary 4.6 P admits a smooth inverse. (if you want to avoid and use $\arctan 2$ instead, note that $(x, y, z) \mapsto (\arctan(y, x), \arccos z)$ is a smooth map defined on an open neighbourhood of $\mathbb{S}^2 \subset \mathbb{R}^3$, so it is smooth when restricted to $\mathbb{S}^2$)

(4) Let $\gamma : \mathbb{R} \to \mathbb{S}^2$ be the smooth map defined as $\gamma(t) = (\cos t, \sin t, 0)$. Compute

$$\int_0^{2\pi n} \left\|\gamma'(t)\right\|_{g_r} dt, \quad \gamma'(t) = \gamma_*(d/dt|_t) \quad \left\|\gamma'(t)\right\|_{g_r} = \sqrt{g_r(\gamma'(t), \gamma'(t))}, \quad n \in \mathbb{N}.$$

Solution. Again, we can simply consider that γ is a map to $\mathbb{R}^3$ and compute instead $\|\gamma'(t)\|_g = \sqrt{g(\gamma'(t), \gamma'(t))}$. The lengthy argument is that

$$g_r(\gamma'(t), \gamma'(t)) = i^*g(\gamma'(t), \gamma'(t)) = g(i_*\gamma'(t), i_*\gamma'(t)) = g((i \circ \gamma)'(t), (i \circ \gamma)'(t))$$

and $i \circ \gamma$ is simply γ as a map $\mathbb{R} \to \mathbb{R}^3$. In these terms, we have

$$\gamma'(t) = -\sin t\partial_x + \cos_t \partial_y,$$

and so

$$\left\|\gamma'(t)\right\|_g = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

We then have

$$\int_0^{2\pi n} \left\|\gamma'(t)\right\|_g dt = 2\pi n.$$

(5) The Poincaré half-plane $\mathbf{H}$ is $\mathbb{R}^2 \cap \{y > 0\}$ as a manifold and comes equipped with the Riemannian metric $g = (dx^{\otimes 2} + dy^{\otimes 2})/y^2$.

(a) Let $p = (x, y)$, $q = (x', y')$ be two points of $\mathbf{H}$. Compute the length $\int_0^1 \|\gamma'(t)\|_g \, dt$ where γ joins $\gamma(0) = p$ with $\gamma(1) = q$ by a straight line: for $0 \le t \le 1$, $\gamma(t) = tq + (1-t)p$.

(b) Same question but now γ is a demi-circle containing p, q and whose centre is on the x-axis.

(c) Denote $z = x + iy$ and consider a map $F : \mathbf{H} \to \mathbb{C} = \mathbb{R}^2$, $F(z) = (z-i)/(z+i) = X + iY$ (Cayley transform). Describe its image and compute $F^*(dX \otimes dX + dY \otimes dY)$. Make a connection with your preceding computations.

6. Differential forms

(1) Consider the following differential forms on $\mathbb{R}^3$:

$$\alpha = x\,dx + y\,dy + z\,dz, \quad \beta = z\,dx + x\,dy + y\,dz, \quad \gamma = xy\,dz.$$

(a) Compute $\alpha \wedge \beta, \alpha \wedge \gamma, \gamma \wedge \beta, (\alpha + \beta) \wedge (\alpha + \gamma)$.
(b) Compute $d\alpha$, $d\beta$, $d\gamma$, $d(\alpha \wedge \beta)$, $d(f\alpha)$, where $f(x, y, z) = xyz$.

Solution. This is a last year's exercise (IV.7.1), its solution can be found in a separate file on moodle.

(2) We continue to consider differential forms on $\mathbb{R}^3$.

(a) Check that the following 1-forms on $\mathbb{R}^3$ are *exact*, that is, can be expressed as df for some $f \in C^\infty(\mathbb{R}^3)$:

$$\eta_1 = dx + 2y\,dy + 3z^2\,dz, \quad \eta_2 = zy\cos(xy)\,dx + zx\cos(xy)\,dy + \sin(xy)\,dz.$$

(b) Show that a 2-form on $\mathbb{R}^3$ with constant function coefficients in the canonical form basis is exact, that is expressible as $d\eta$ for $\eta \in \Lambda^1(\mathbb{R}^3)$.
(c) Find a solution to $d\xi = \omega$, where $\omega = z\,dx \wedge dz + dy \wedge dz$.
(d) Find all 1-forms η such that $\eta \wedge dz = 0$.

Solution. This is, again, last year (IV.7.2-IV.7.5).

(3) On $\mathbb{R}^3$, consider $\omega = dx \wedge dy \wedge dz$, the standard volume form. Let $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$ a vector field on $\mathbb{R}^3$. A contraction of ω with V is denoted $\iota_V\omega$ and is defined as

$$\iota_V\omega(X, Y) := \omega(V, X, Y), \quad X, Y \in \mathcal{T}(\mathbb{R}^3).$$

(a) Show that ι_V is a differential 2-form and compute its coefficients in the basis $dx \wedge dy, dx \wedge dz, dy \wedge dz$.

Solution. The function $(X, Y) \mapsto \omega(V, X, Y)$ is C^∞-bilinear and antisymmetric, so according to lectures it is a differential 2-form. For any 2-form $\eta = f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$, we know that $f = \eta(\partial_x, \partial_y), g = \eta(\partial_x, \partial_z), h = (\partial_y, \partial_z)$. Thus we need to compute $\iota_V\omega(\partial_x, \partial_y), \iota_V\omega(\partial_x, \partial_z), \iota_V\omega(\partial_y, \partial_z)$. To do this, we note that $\omega(\partial_x, \partial_y, \partial_z) = 1$ and that if any basis vector repeats, we get zero:

$$\begin{aligned}\iota_V\omega(\partial_x, \partial_y) &= \omega(V, \partial_x, \partial_y) = \omega(V_x\partial_x, \partial_x, \partial_y) + \omega(V_y\partial_y, \partial_x, \partial_y) + \omega(V_z\partial_z, \partial_x, \partial_y)\\ &= \omega(V_z\partial_z, \partial_x, \partial_y) = \omega(\partial_x, \partial_y, V_z\partial_z) = V_z\omega(\partial_x, \partial_y, \partial_z) = V_z.\end{aligned}$$

In the fourth expression, we only kept $V_z\partial_z$ since all two other terms produce zeros by antisymmetry. We then shuffled the terms. Similarly,

$$\begin{aligned}\iota_V\omega(\partial_x, \partial_z) &= \omega(V, \partial_x, \partial_z) = \omega(V_x\partial_x, \partial_x, \partial_z) + \omega(V_y\partial_y, \partial_x, \partial_z) + \omega(V_z\partial_z, \partial_x, \partial_z)\\ &= \omega(V_y\partial_y, \partial_x, \partial_z) = -\omega(\partial_x, V_y\partial_y, \partial_z) = -V_y\omega(\partial_x, \partial_y, \partial_z) = -V_y.\end{aligned}$$

$$\begin{aligned}\iota_V\omega(\partial_y, \partial_z) &= \omega(V, \partial_y, \partial_z) = \omega(V_x\partial_x, \partial_y, \partial_z) + \omega(V_y\partial_y, \partial_y, \partial_z) + \omega(V_z\partial_z, \partial_y, \partial_z)\\ &= \omega(V_x\partial_x, \partial_y, \partial_z) = V_x\omega(\partial_x, \partial_y, \partial_z) = V_x.\end{aligned}$$

The formula is thus

$$\iota_V\omega = V_z\,dx \wedge dy - V_y\,dx \wedge dz + V_x\,dy \wedge dz.$$

(b) Let $W = W_x\partial_x + W_y\partial_y + W_z\partial_z$ be a vector field. We consider also $\eta = W_x\,dx + W_y\,dy + W_z\,dz$. Find V such that $\iota_V\omega = d\eta$.
Indication. The formula of the previous exercise together with the example of the lectures suggest that $V = \nabla \times W$. Reproduce the computation if unsure!

(4) The spherical area form ω_E is defined as

$$\omega_E := \iota_E\omega, \quad E = x\partial_x + y\partial_y + z\partial_z.$$

The intuition is that we "remove" the radial direction of ω via the contraction with the Euler vector field E.

(a) Find the explicit formula for ω_E.
Solution. Thanks to the computations above, $\omega_E = z\,dx \wedge dy - y\,dx \wedge dz + x\,dy \wedge dz$. Here, we denote by x the function $(x, y, z) \mapsto x$ and so on.

(b) Compute $P^*(\omega_E)$ where $P : \mathbb{R}^2 \to \mathbb{R}^3$ is the familiar spherical coordinates map $P(\varphi, \theta) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$.
Solution. Writing $P = (P^1, P^2, P^3)$ we note that $P^*x = P^1$, $P^*y = P^2$, $P^*z = P^3$ and so we have

$$P^*\omega_E = P^3\,dP^1 \wedge dP^2 - P^2\,dP^1 \wedge dP^3 + P^1\,dP^2 \wedge dP^3.$$

We have already computed that

$$dP^1 = -\sin\varphi\sin\theta d\varphi + \cos\varphi\cos\theta d\theta,\ dP^2 = \cos\varphi\sin\theta d\varphi + \sin\varphi\cos\theta d\theta,$$

$$dP^3 = -\sin\theta d\theta.$$

So we now have to compute the double products. Let me write c_θ for the function $\theta \mapsto \cos\theta$, s_θ for the function $\theta \mapsto \sin_\theta$ and similarly for φ. Then:

$$\begin{aligned} P^3\,dP^1 \wedge dP^2 &= c_\theta(-s_\varphi s_\theta d\varphi + c_\varphi c_\theta d\theta) \wedge (c_\varphi s_\theta d\varphi + s_\varphi c_\theta d\theta) \\ &= c_\theta(-s_\varphi s_\theta s_\varphi c_\theta - c_\varphi c_\theta c_\varphi s_\theta) d\varphi \wedge d\theta = -s_\theta c_\theta^2\,d\varphi \wedge d\theta. \end{aligned}$$

$$-P^2\,dP^1 \wedge dP^3 = -s_\varphi s_\theta(-s_\varphi s_\theta d\varphi + c_\varphi c_\theta d\theta) \wedge (-s_\theta d\theta) = -s_\varphi^2 s_\theta^3 d\varphi \wedge d\theta.$$

$$P^1\,dP^2 \wedge dP^3 = c_\varphi s_\theta(c_\varphi s_\theta d\varphi + s_\varphi c_\theta d\theta) \wedge (-s_\theta d\theta) = -c_\varphi^2 s_\theta^3 d\varphi \wedge d\theta.$$

The end result is

$$P^*\omega_E = -(s_\theta c_\theta^2 + s_\varphi^2 s_\theta^3 + c_\varphi^2 s_\theta^3) d\varphi \wedge d\theta = -s_\theta\,d\varphi \wedge d\theta = s_\theta d\theta \wedge d\varphi.$$

Two comments. First, from the viewpoint of its sign it seems more natural to treat θ as the first coordinate, that is, first specify the altitude, and then the longitude. Formally it corresponds to choosing $d\theta \wedge d\varphi$ as the preferred orientation of (φ, θ). Second, the factor $\sin\theta$ corresponds, intuitively, to the fact that the elementary volume is not a "tiny square in θ, φ" but a tiny square with a factor that changes with position of the angle. But I leave your intuition to you.

(c) Compute $(\sigma^{-1})^* i^* \omega_E$, where $i : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map, and σ^{-1} is the inverse of the stereographic projection $\sigma(x, y, z) = \frac{1}{1-z}(x, y)$.

Solution. We are computing $F^*\omega_E$, where

$$F(u,v) = (F^1(u,v), F^2(u,v), F^3(u,v)) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

and as we saw in the previous file,

$$dF^1 = \frac{2-2u^2+2v^2}{(1+u^2+v^2)^2}\,du - \frac{4uv}{(1+u^2+v^2)^2}\,dv$$

$$dF^2 = -\frac{4uv}{(1+u^2+v^2)^2}\,du + \frac{2+2u^2-2v^2}{(1+u^2+v^2)^2}\,dv, \quad dF^3 = \frac{4u}{(1+u^2+v^2)^2}\,du + \frac{4v}{(1+u^2+v^2)^2}\,dv.$$

As before, given two 1-forms on $\mathbb{R}^2$, $\eta = a\,du + b\,dv$, $\eta' = c\,du + d\,dv$, their wedge product is $\eta \wedge \eta' = (ad - bc)\,du \wedge dv$. We then compute

$$\begin{aligned} dF^1 \wedge dF^2 &= \left(\frac{2-2u^2+2v^2}{(1+u^2+v^2)^2} \cdot \frac{2+2u^2-2v^2}{(1+u^2+v^2)^2} - \frac{16u^2v^2}{(1+u^2+v^2)^4}\right) du \wedge dv \\ &= \frac{4(1-(u^2-v^2)^2) - 16u^2v^2}{(1+u^2+v^2)^4}\,du \wedge dv \\ &= \frac{4(1-(u^2+v^2)^2)}{(1+u^2+v^2)^4}\,du \wedge dv = \frac{4(1-u^2-v^2)}{(1+u^2+v^2)^3}\,du \wedge dv. \end{aligned}$$

$$\begin{aligned} dF^1 \wedge dF^3 &= \left(\frac{2-2u^2+2v^2}{(1+u^2+v^2)^2} \cdot \frac{4v}{(1+u^2+v^2)^2} + \frac{16u^2v}{(1+u^2+v^2)^4}\right) du \wedge dv \\ &= \frac{8v(1-u^2+v^2+2u^2)}{(1+u^2+v^2)^4}\,du \wedge dv = \frac{8v}{(1+u^2+v^2)^3}\,du \wedge dv. \end{aligned}$$

$$\begin{aligned} dF^2 \wedge dF^3 &= \left(-\frac{16uv^2}{(1+u^2+v^2)^4} - \frac{4u(2+2u^2-2v^2)}{(1+u^2+v^2)^4}\right) du \wedge dv \\ &= -\frac{8u}{(1+u^2+v^2)^3}\,du \wedge dv. \end{aligned}$$

There are many symmetries in these computations, so even if the expressions are a bit heavy, the steps are not hard and the eventual results look relatively neat. Finally,

$$\begin{aligned} F^*\omega_E &= F^3\,dF^1 \wedge dF^2 - F^2\,dF^1 \wedge dF^3 + F^1\,dF^2 \wedge dF^3 \\ &= \left(-\frac{4(1-u^2-v^2)^2}{(1+u^2+v^2)^4} - \frac{16v^2}{(1+u^2+v^2)^4} - \frac{16u^2}{(1+u^2+v^2)^4}\right) du \wedge dv \\ &= -\frac{4}{(1+u^2+v^2)^2}\,du \wedge dv = \frac{4}{(1+u^2+v^2)^2}\,dv \wedge du. \end{aligned}$$

We see that the coefficient before $du \wedge dv$ is negative, meaning that the map σ^{-1} is in fact orientation-reversing (see for instance `https://tinyurl.com/4j9j4vzz`).

(5) We consider $\mathbb{R}^4$ whose points we write as (t, x, y, z).

(a) List all the basis forms in each dimension.

(b) For any basis form $\omega \in \Lambda^k(\mathbb{R}^4)$, define its Hodge dual as the unique form $*\omega$ such that $\omega \wedge *\omega = (-1)^\omega dt \wedge dx \wedge dy \wedge dz$, where $(-1)^\omega = -1$ if ω does not contain dt, and $+1$ otherwise. Find $*\eta$ for all the basis forms.

Solution. That does not seem too hard of a task.

- Zero-forms have only the constant function 1 as a basis zero-form. It does not contain dt, so $*1 = -dt \wedge dx \wedge dy \wedge dz$.
- One-forms have as basis the four forms dt, dx, dy, dz. We see that

$$*dt = dx \wedge dy \wedge dz, \quad *dx = dt \wedge dy \wedge dz,$$

$$*dy = -dt \wedge dx \wedge dz, \quad *dz = dt \wedge dx \wedge dy.$$

- Two-forms have six basis forms. We see that

$$*(dt \wedge dx) = dy \wedge dz, \quad *(dt \wedge dy) = -dx \wedge dz, \quad *(dt \wedge dz) = dx \wedge dy,$$

$$*(dx \wedge dy) = -dt \wedge dz, \quad *(dx \wedge dz) = dt \wedge dy, \quad *(dy \wedge dz) = -dt \wedge dx.$$

  My signs are different from the wikipedia page on Hodge star which seems to contradict itself, oh well.
- Three-forms have four basis forms. We see that

$$*(dt \wedge dx \wedge dy) = dz, \quad *(dt \wedge dy \wedge dz) = dx$$

$$*(dt \wedge dx \wedge dz) = -dy, \quad *(dx \wedge dy \wedge dz) = dt.$$

- $*(dt \wedge dx \wedge dy \wedge dz) = 1$.

(c) We extend $*$ on all forms by linearity, putting $* \sum f_\alpha \omega_\alpha = \sum f_\alpha * \omega_\alpha$ where ω_α are the basis forms. Compute $** \eta$ for any k-form η.
Solution. What happens on basis forms? If η is a 1 or 3-basis form, then from the computation above, we see that $** \eta = \eta$. On the other hand, for $0, 2, 4$-basis forms, we see that $** \eta = -\eta$.
In general, $** \sum f_\alpha \omega_\alpha = \sum f_\alpha ** \omega_\alpha$. Thus for $\eta \in \Lambda^k(\mathbb{R}^4)$, we have $** \eta = (-1)^{k+1}\eta$.

(d) Let $f \in C^\infty(\mathbb{R}^4)$. Interpret the equation $*d*df = 0$.
Solution. The expression makes sense as a composition of linear maps d and $*$. We have $df = f_t\, dt + f_x\, dx + f_y\, dy + f_z\, dz$, where the subindex denotes the partial derivative. Thus

$$*df = f_t\, dx \wedge dy \wedge dz + f_x\, dt \wedge dy \wedge dz - f_y\, dt \wedge dx \wedge dz + f_z\, dt \wedge dx \wedge dy,$$

$$d*df = (f_{tt} - f_{xx} - f_{yy} - f_{zz})\, dt \wedge dx \wedge dy \wedge dz.$$

This allows to conclude that

$$*d*df = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) f = \Box f$$

where $\Box$ is the famous d'Alembert operator (for $c = 1$ lightspeed). In other words, $*d*df = 0$ is the wave equation.

(e) We consider $F \in \Lambda^2(\mathbb{R}^4)$, $J \in \Lambda^1(\mathbb{R}^4)$, that we conveniently write as

$$F = -E_1\, dt \wedge dx - E_2\, dt \wedge dy - E_3\, dt \wedge dz + B_1 dy \wedge dz - B_2\, dx \wedge dz + B_3\, dx \wedge dy,$$

$$J = -\rho\, dt + J_1\, dx + J_2\, dy + J_3\, dz.$$

Interpret the equations $dF = 0$ and $d*F = 4\pi * J$.
Solution. We have

$$\begin{aligned} dF = &(-\partial_y E_1 + \partial_x E_2 + \partial_t B_3) dt \wedge dx \wedge dy \\ &+ (-\partial_z E_1 + \partial_x E_3 - \partial_t B_2) dt \wedge dx \wedge dz \\ &+ (-\partial_z E_2 + \partial_y E_3 + \partial_t B_1) dt \wedge dy \wedge dz \\ &+ (\partial_x B_1 + \partial_y B_2 + \partial_z B_3) dx \wedge dy \wedge dz. \end{aligned}$$

Write the following vector fields: $E = E_1\partial_x + E_2\partial_y + E_3\partial_z$, $B = B_1\partial_x + B_2\partial_y + B_3\partial_z$. Recall that the curl is defined as

$$\nabla \times E = (\partial_y E_3 - \partial_z E_2)\partial_x + (\partial_z E_1 - \partial_x E_3)\partial_y + (\partial_x E_2 - \partial_y E_1)\partial_z.$$

If we introduce the vector field $\frac{\partial B}{\partial t} = \partial_t B_1\,\partial_x + \partial_t B_2\,\partial_y + \partial_t B_3\,\partial_z$ then $dF = 0$ becomes

$$\nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \nabla B = 0$$

where $\nabla B = \partial_x B_1 + \partial_y B_2 + \partial_z B_3$ is the divergence. This is the first pair of Maxwell equations.

We now proceed to compute

$$*F = -E_1\,dy \wedge dz + E_2\,dx \wedge dz - E_3\,dx \wedge dy - B_1\,dt \wedge dx - B_2\,dt \wedge dy - B_3\,dt \wedge dz,$$

$$\begin{aligned} d * F &= (-\partial_t E_3 - \partial_y B_1 + \partial_x B_2)\,dt \wedge dx \wedge dy \\ &+ (\partial_1 E_2 - \partial_z B_1 + \partial_x B_3)\,dt \wedge dx \wedge dz \\ &+ (-\partial_t E_1 - \partial_z B_2 + \partial_y B_3)\,dt \wedge dy \wedge dz \\ &+ (-\partial_x E_1 - \partial_y E_2 - \partial_z E_3)\,dx \wedge dy \wedge dz. \end{aligned}$$

Finally,

$$\begin{aligned} 4\pi * J &= -4\pi\rho\,dx \wedge dy \wedge dz + 4\pi J_1\,dt \wedge dy \wedge dz \\ &- 4\pi J_2\,dt \wedge dx \wedge dz + 4\pi J_3\,dt \wedge dx \wedge dy. \end{aligned}$$

Introducing, again, the notations

$$\nabla \times B = (\partial_y B_3 - \partial_z B_2)\partial_x + (\partial_z B_1 - \partial_x B_3)\partial_y + (\partial_x B_2 - \partial_y B_1)\partial_z$$

$$\frac{\partial E}{\partial t} = \partial_t E_1\,\partial_x + \partial_t E_2\,\partial_y + \partial_t E_3\partial_z, \quad \nabla E = \partial_x E_1 + \partial_y E_2 + \partial_z E_3$$

we see that $d * F = 4\pi * J$ becomes

$$\frac{\partial E}{\partial t} - \nabla \times B = -4\pi j, \quad \nabla E = 4\pi\rho$$

where $j = J_1\partial_x + J_2\partial_y + J_3\partial_z$. This is the second pair of Maxwell's equations.
If we were less in a rush, we would have done a more theoretical approach to Hodge. In particular, the signs choice in this exercise may seem completely ad-hoc. It is not, and is related to the Minkowski metric $dt \otimes dt - dx \otimes dx - dy \otimes dy - dz \otimes dz$.

(f) From Poincaré's lemma we know that $dF = 0 \Leftrightarrow F = dA$ for some $A = A_0 dt + A_1 dx + A_2 dy + A_3 dz$. What becomes of the second equation if we rewrite it using A?
Indication. You will get the potential form of Maxwell's equations. Perhaps fixing a gauge is a nice thing to do.

Homework problem set 1

20 pts max, the exercises offer slightly more. It may look long, but it is in fact a step-by-step guide! Feel free to ask questions.

1. Reminders on linear algebra. Let V be a vector space over $\mathbb{R}$ and $f_1, \dots, f_n$ denote a basis in the dual vector space V^* (in particular $\dim V = n$).

(1) (1pt) Let $b : V \times V \to \mathbb{R}$ be a bilinear form (refer to MAA201). Prove that b can be written as $b = s + a$ where $s : V \times V \to \mathbb{R}$ is bilinear and symmetric, $s(v, w) = s(w, v)$, and $a : V \times V \to \mathbb{R}$ is antisymmetric, $a(v, w) = -a(w, v)$.

Solution. $s(v, w) = (b(v, w) + b(w, v))/2$, $a(v, w) = (b(v, w) - b(w, v))/2$.

(2) (1pt) Denote $\mathcal{B}(V, \mathbb{R})$ the set of all bilinear forms. Show that it is a vector subspace (with respect to the addition and $\mathbb{R}$-multiplication defined valuewise).

Solution. The set of functions $V \times V \to \mathbb{R}$ is a vector space with addition and scalar multiplication defined pointwise. If b_1, b_2 are two bilinear forms then $\lambda b_1 + \mu b_2$ is again bilinear, for

$$\begin{aligned}(\lambda b_1 + \mu b_2)(av_1 + bv_2, w) &:= \lambda b_1(av_1 + bv_2, w) + \mu b_2(av_1 + bv_2, w)\\ &= a\lambda b_1(v_1, w) + b\lambda b_1(v_2, w) + a\mu b_2(v_1, w) + b\mu b_2(v_2, w)\\ &= a(\lambda b_1 + \mu b_2)(v_1, w) + b(\lambda b_1 + \mu b_2)(v_2, w),\end{aligned}$$

and the similar verification for the second argument.

(3) (3pt) For two $l_1, l_2 \in V^*$, denote

$$l_1 \otimes l_2 : V \times V \to \mathbb{R}, \quad l_1 \otimes l_2(v, w) := l_1(v) \cdot l_2(w).$$

Prove that $l_1 \otimes l_2 \in \mathcal{B}(V, \mathbb{R})$, that $\mathsf{Span}(f_i \otimes f_j)_{i,j} = \mathcal{B}(V, \mathbb{R})$. Compute $\dim \mathcal{B}(V, \mathbb{R})$.

Solution. For the first point, we use the linearity of l_1, l_2 as follows:

$$\begin{aligned}l_1 \otimes l_2(av_1 + bv_2, w) &= l_1(av_1 + bv_2)l_2(w) = (al_1(v_1) + bl_1(v_2))l_2(w)\\ &= al_1(v_1)l_2(w) + bl_1(v_2)l_2(w) = a(l_1 \otimes l_2(v_1, w)) + b(l_1 \otimes l_2(v_2, w)).\end{aligned}$$

For the second, there are many ways to do this, here is one that uses matrices. Denote $(e_1, \dots, e_n)$ the pre-dual basis of $(f_1, \dots, f_n)$. For any bilinear form b, denote $M(b)_{ij} = b(e_i, e_j)$. The assignement $b \mapsto M(b)$ is an isomorphism of vector spaces

$$M : \mathcal{B}(V, \mathbb{R}) \xrightarrow{\sim} \mathsf{Mat}_n(\mathbb{R}).$$

Strictly speaking that is quadratic forms course material, so it is immediate. But if we were to prove this, we could simply try the following inverse:

$$A \in \mathsf{Mat}_n(\mathbb{R}), \quad \mapsto \quad B(A) := \sum_{ij} A_{ij} f_i \otimes f_j.$$

Note that $f_i \otimes f_j(e_k, e_m) = f_i(e_k) f_j(e_m) = \delta_{ik}\delta_{jm}$. Because of this, if $A \in \mathsf{Mat}_n(\mathbb{R})$, then

$$M(B(A))_{km} = B(A)(e_k, e_m) = \sum_{ij} A_{ij} f_i(e_k) f_j(e_m) = \sum_{ij} A_{ij}\delta_{ik}\delta_{jm} = A_{km}.$$

On the other hand, by bilinearity, if $v = \sum v_i e_i$ and $w = \sum w_j e_j$, then

$$b(v, w) = \sum_{ij} b(e_i, e_j) v_i w_j = \sum_{ij} M(b)_{ij} v_i w_j.$$

and thus

$$B(M(b))(v,w) = \sum_{ij} M(b)_i j f_i(v) f_j(w) = \sum_{ij} b(e_i, e_j) v_i w_j = b(v,w).$$

The dimension of $\mathcal{B}(V, \mathbb{R})$ is thus n^2. Finally, the family $\{f_i \otimes f_j\}_{i,j}$ is a basis since $M(f_i \otimes f_j)$ is the nxn matrix of zeros everywhere except at the ij-th place. Such matrices form a basis of $Mat_n(\mathbb{R})$.

(4) (1pt) Denote $\mathcal{A}(V, \mathbb{R})$ the set of all antisymmetric bilinear forms. Show that it is a vector subspace of $\mathcal{B}(V, \mathbb{R})$.

Solution.

$$\lambda a(v,w) + \mu a'(v,w) = \lambda(-a(w,v)) + \mu(-a'(w,v)) = -(\lambda a(w,v) + \mu a'(w,v)).$$

(5) (2pt) For two $l_1, l_2 \in V^*$, denote

$$l_1 \wedge l_2 : V \times V \to \mathbb{R}, \quad l_1 \wedge l_2(v,w) := l_1(v) \cdot l_2(w) - l_2(v) \cdot l_1(w).$$

Prove that $l_1 \wedge l_2 \in \mathcal{A}(V, \mathbb{R})$ and that $\mathsf{Span}(f_i \wedge f_j)_{i,j} = \mathcal{A}(V, \mathbb{R})$.

Solution. $l_1 \wedge l_2$ is two times the antisymmetric part of the bilinear form $l_1 \otimes l_2$. For the rest, see below.

(6) (2pt) What is the dimension of $\mathcal{A}(V, \mathbb{R})$ and how to write a basis in it using $f_i \wedge f_j$?

Solution. We constructed the isomorphism $M : \mathcal{B}(V, \mathbb{R}) \xrightarrow{\sim} \mathsf{Mat}_n(\mathbb{R})$. It is easy to see that the image $M(\mathcal{A}(V, \mathbb{R}))$ corresponds to antisymmetric matrices that I will denote $Mat_n^A(\mathbb{R})$. It is of course a subspace of $Mat_n(\mathbb{R})$ (trivial check, or direct corollary of our isomorphism).

The space $\mathsf{Mat}_n^A(\mathbb{R})$ has the matrices E^{ij} such that

$$(E^{ij})_{ij} = 1, (E^{ij})_{ji} = -1, \text{ else } (E^{ij})_{km} = 0.$$

Any matrix $A \in \mathsf{Mat}_n^A(\mathbb{R})$ is uniquely determined by its matrix coefficients A_{ij} for $1 \leq i < j \leq n$ (upper diagonal). We can indeed write

$$A = \sum_{1 \leq i < j \leq n} A_{ij} E^{ij}$$

since that corresponds to filling up the upper diagonal with A_{ij} and the lower diagonal with $-A_{ij}$. A cleaner way to say this: one has $E^{ij} = I^{ij} - I^{ji}$ where I^{ij} is the matrix of the canonical basis for $\mathsf{Mat}_n(\mathbb{R})$. And since $A_{ii} = 0$ and $A_{ij} = -A_{ji}$,

$$\sum_{1 \leq i < j \leq n} A_{ij} E^{ij} = \sum_{1 \leq i < j \leq n} A_{ij}(I^{ij} - I^{ji}) = \sum_{1 \leq i, j \leq n} A_{ij} I^{ij} = A.$$

So the family $\{E^{ij}\}_{1 \leq i < j \leq n}$ is generating. It is also a basis: if $\sum_{1 \leq i < j \leq n} \lambda_{ij} E^{ij} = 0$ then, by defining $\lambda_{ji} = -\lambda_{ij}$ and $\lambda_{ii} = 0$, we see that the associated matrix $A = \sum \lambda_{ij} I^{ij}$ is zero. The dimension of $\mathsf{Mat}_n^A(\mathbb{R}) \cong \mathcal{A}(V, \mathbb{R})$ is thus $n(n-1)/2$.

The only thing that remains to observe is that

$$f_i \wedge f_j(e_i, e_j) = 1 - 0, f_i \wedge f_j(e_j, e_i) = 0 - 1, \text{ else } f_i \wedge f_j(e_k, e_m) = 0$$

so $M(f_i \wedge f_j) = E^{ij}$.

2. More on torus. Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ and denote

$$q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2, \quad v = (x, y) \mapsto [v] \equiv [(x, y)]$$

the quotient map. The goal of this exercise is to show how one can go from $\mathbb{T}$ back to $\mathbb{R}^2$, at least, locally.

(1) (1pt) Fix $v \in \mathbb{R}^2$, and define

$$U_v := B(v, 1/3) = \{w \in \mathbb{R}^2 \quad | \quad \|w - v\| < 1/3\}.$$

Show that for any $(n, m) \in \mathbb{Z}^2 \setminus 0$, one has $U_v \cap U_{v+(n,m)} = \emptyset$.

Solution. Write $v = (v_1, v_2)$. Every U_v sits inside the square

$$Sq_v := \{(x, y)|x \in]-1/3 + v_1, 1/3 + v_1[, y \in]-1/3 + v_2, 1/3 + v_2[.$$

Indeed, the equality $(x-v_1)^2+(y-v_2)^2 = 1/9$ implies that $|x-v_1| < 1/3$ and $|y-v_2| < 1/3$. Can theses squares intersect? For (x, y) to belong both to Sq_v and $Sq_{v+(n,m)}$, the following needs to be satisfied:

$$x \in]-1/3 + v_1, 1/3 + v_1[\cap] - 1/3 + v_1 + n, 1/3 + v_1 + n[,$$
$$y \in]-1/3 + v_2, 1/3 + v_2[\cap] - 1/3 + v_2 + m, 1/3 + v_2 + m[.$$

This is impossible since all four intervals have length 2/3 and moving any interval of length < 1 by an integer creates a disjoint interval.

(2) (2pt) Denote $S_v := q^{-1}(q(U_v))$. Show that

$$S_v = \sqcup_{(n,m)\in\mathbb{Z}^2} U_{v+(n,m)}$$

(it may be helpful to draw pictures) and use this to conclude that $q(U_v)$ is open in $\mathbb{T}$.

Solution. $(x, y) \in S_v$ implies that $[(x, y)] \in q(U_v)$, meaning that $(x, y) = (x' + n, y' + m)$ for (x', y') in U_v. On the other hand, any point of the form $(x' + n, y' + m)$ projects to $[(x', y')]$. S_v is a union of countable amount of open balls, it is open, and so is $q(U_v)$ by the definition of quotient topology.

(3) (3pt) Consider the assignment

$$(x, y) \in U_{v+(n,m)} \quad \mapsto \quad \sigma_v(x, y) = (x - n, y - m).$$

Prove that this defines a continuous map $\sigma_v : S_v \to U_v$ (you can use either sequential arguments from MAA202, or verify by taking inverse images of opens).

Solution. The formula given above is the restriction of σ_v to $U_{v+(n,m)}$. This formula is a translation, an affine map, so it is continuous. Thus $\sigma_v|_{(U_{v+(n,m)}}$ is continuous for each (n, m). Since $U_{v+(n,m)}$ cover S_v and continuous functions form a sheaf, we conclude.

(4) (2pt) We consider S_v as a topological space on its own, and put on it the equivalence relation

$$w_1, w_2 \in S_v, \quad w_1 \sim w_2 \quad \iff \quad w_1 - w_2 \in \mathbb{Z}^2.$$

Prove that the composition $S_v \hookrightarrow \mathbb{R}^2 \to \mathbb{T}^2$ induces $S_v/_\sim = q(U_v)$ (each equivalence class in $S_v/_\sim$ is present in $q(U_v)$ and vice versa).

Solution. This is a badly formulated exercise so let me rephrase it better. The claim that I want to verify is that $q(S_v) = q(U_v)$ and that the map $q|_{S_v} : S_v \to q(U_v)$ satisfies the universal property of the quotient, with S_v viewed as a topological space equipped with an equivalence relation.

The first point is addressed as follows. $q(S_v) = q(q^{-1}(q(U_v))$ translates into

$$q(S_v) = \{p \in \mathbb{T} | \exists y : p = q(y) \text{ and } q(y) \in q(U_v)\}.$$

This means that $q(S_v) \subset q(U_v)$. On the other hand, since $U_v \subset S_v$, we have $q(U_v) \subset q(S_v)$.

As for the second point, it is really one big tautology. Note that $q|_{S_v}$ is continuous as a restriction of continuous map. Also note that the map $q|_{S_v}$ sends equivalent points to equal points in $q(U_v)$.

Given any continuous $f : S_v \to T$ that identifies equivalent points, we construct $\tilde{f} : q(U_v) \to T$ by declaring $\tilde{f}([x, y]) = f(x, y)$. It obviously respects equivalent points. We also note that $\tilde{f} \circ q|_{S_v}$ is f. Since $f^{-1}(V) = q^{-1}\tilde{f}^{-1}$, we have that $\tilde{f}^{-1}$ is continuous just like in the case of quotient.

This tautological verification is the statement that can be rephrased into "a restriction of a quotient map is a quotient map".

(5) (2pt) Prove that $\sigma_v : S_v \to U_v$ respects the equivalence relation: $w_1 \sim w_2 \implies \sigma_v(w_1) = \sigma_v(w_2)$.

Solution. Write $w_1 = a + v + (n, m)$ and $w_2 = b + v + (n', m')$. In this presentation $\|a\| < 1/3$, $\|b\| < 1/3$. Then $w_1 - w_2 = a - b + (n - n', m - m')$. Since $w_1 \sim w_2$ this means that $a - b \in \mathbb{Z}^2$, but due to the norm restrictions it can only happen if $a = b$. Finally

$$\sigma_v(w_1) = \sigma_v(a + v + (n, m)) = \sigma_v(a) = \sigma_v(b) = \sigma_v(b + v + (n', m')) = \sigma_v(w_2).$$

(6) (2pt) Us the preceding results to show that there exists a continuous map $s_v : q(U_v) \to U_v$ such that $q \circ s_v$ is the identity.

Solution. The map s_v is the factorisation of σ_v as $s_v \circ q|_{S_v} = \sigma_v$. It evidently satisfies $s_v \circ q|_{U_v} = \sigma_v|_{U_v} = \mathrm{id}_{U_v}$. But we are asked to check the other composition, that should be written as $q|_{U_v} \circ s_v$.

Note that $q|_{U_v} \circ \sigma_v = q|_{S_v}$, since σ_v merely removes some integers from the coordinates. Thus $q|_{U_v} \circ s_v \circ q|_{S_v} = q|_{S_v}$. Restricting it to U_v, we get $q|_{U_v} \circ s_v \circ q|_{U_v} = q|_{U_v}$. Let $p \in q(U_v)$. This means that there exists $(x, y) \in U_v$ with $q(x, y) = p$. But then

$$p = q(x, y) = q(s_v(q(x, y))) = q(s_v(p)).$$

In fact, we have proven that $\mathbb{T}$ is locally homeomorphic to some U_v, meaning that it is a topological manifold (for Hausdorff, see comment below).

(7) (1pt) What breaks down in the above argument if instead of $B(v, 1/3)$ we consider $B(v, 2/3)$?

Solution. Well, balls intersect. There will be ambiguity in defining σ_v for example.

We can modify it differently though, by replacing U_v with $U_v(\varepsilon) = B(v, \varepsilon)$ with $\varepsilon < 1/2$. For each two different points $p_1, p_2 \in \mathbb{T}$ we can find $q(U_{v_1}(\varepsilon_1)) \ni p_1$ and $q(U_{v_2}(\varepsilon_2)) \ni p_2$ that do not intersect. Indeed, pick any v_i with $q(v_i) = p_i$ and so that they are contained in $[0, 1[\times[0, 1[$ (if your v_i is not like that, take fractional part of its coordinates).

Next, due to Hausdorff property of $\mathbb{R}^2$, there exist open balls $U_{v_1}(\varepsilon_1)$, $U_{v_2}(\varepsilon_2)$ that do not intersect. Can their q-images intersect. If we take $q^{-1}q$-sets $S_{v_1}(\varepsilon_1)$ $S_{v_2}(\varepsilon_2)$, they cannot intersect either. Indeed, let $u \in U_{v_1+(n,m)}(\varepsilon_1) \cap U_{v_2+(n',m')}(\varepsilon_2)$. If $\varepsilon_1, \varepsilon_2$ are sufficiently small the intersection is nonempty iff $(n, m) = (n', m')$ (otherwise balls will be to far away). But then we can take $u - (n, m)$ that will belong both to $U_{v_1}(\varepsilon_1)$ and $U_{v_2}(\varepsilon_2)$.

So $S_{v_1}(\varepsilon_1) \cap S_{v_2}(\varepsilon_2) = \emptyset$. And if $r_1 \in q(U_{v_1}(\varepsilon_1))$, then all its representatives belong to $S_{v_1}(\varepsilon_1)$, and similarly for $r_2 \in q(U_{v_2}(\varepsilon_2))$. Thus r_1 can never be equal r_2 and $q(U_{v_1}) \cap q(U_{v_2}) = \emptyset$.

Homework problem set 2

20 pts max, the exercises offer slightly more. It may look long, but it is in fact a step-by-step guide! I also give some motivation for the exercises. Feel free to ask questions.

1. Functions on the torus. *In this exercise we shall put a sheaf of $\mathbb{R}$-valued functions on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$. The natural idea is to use the quotient map $q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}$. For $U \in \mathsf{Op}(\mathbb{T})$, define*

$$\mathcal{P}(U) := \{f : U \to \mathbb{R} \quad | \quad q^*(f) = f \circ q \in C^\infty_{\mathbb{R}^2}(q^{-1}(U))\}$$

(1) Let $C^\infty_{per}(\mathbb{R}^2) \subset C^\infty(\mathbb{R}^2)$ consist of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ that are smooth and periodic: $f(x+n, y+m) = f(x, y)$ for all $(n, m) \in \mathbb{Z}^2$.

(a) (2pt) Prove that for $f, g \in C^\infty_{per}(\mathbb{R}^2)$ and $\lambda, \mu \in \mathbb{R}$, the functions $\lambda f + \mu g$ and fg are smooth periodic.

Solution. We write z for vectors of the form (n, m). The periodicity condition says that for all $v \in \mathbb{R}^2$ and $z \in \mathbb{Z}^2$, one has $f(v + z) = f(v)$.

Then $(\lambda f + \mu g)(v + z) = \lambda f(v + z) + \mu g(v + z) = \lambda f(v) + \mu g(v) = (\lambda f + \mu g)(v)$. Similarly, $f \cdot g(v + z) = f(v + z)g(v + z) = f(v)g(v) = f \cdot g(v)$.

(b) (2pt) Prove that for $f \in C^\infty_{per}(\mathbb{R}^2)$, its partial derivatives $\partial_x f$ and $\partial_y f$, viewed as functions $\mathbb{R}^2 \to \mathbb{R}$, are also smooth periodic.

Solution.

$$\partial_x f(x_0 + n, y_0 + m) := \lim_{t\to 0} \frac{f(x_0 + n + t, y_0 + m) - f(x_0 + n, y_0 + m)}{t}$$

$$= \lim_{t\to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} = \partial_x f(x_0, y_0).$$

We can compute similarly for ∂_y.

(2) (3pt) Check that the assignment $f \mapsto q^*(f) = f \circ q$ establishes a bijection

$$q^* : \mathcal{P}(\mathbb{T}) \xrightarrow{\sim} C^\infty_{per}(\mathbb{R}^2).$$

Solution. Note that $q^* f(v + z) = f(q(v + z)) = f(q(v)) = q^* f(v)$, so the map q^* indeed lands in periodic functions. Any smooth periodic map $g : \mathbb{R}^2 \to \mathbb{R}$ induces a map $\bar{g} : \mathbb{T} \to \mathbb{R}$ that satisfies, by the universal property, the equation $g = \bar{g} \circ q = q^*(\bar{g})$. Thus $\bar{g} \in \mathcal{P}(\mathbb{T})$ and q^* is surjective.

The equation $q^* f = 0$ means that for each $v \in \mathbb{R}^2$, one has $f(q(v)) = f([v]) = 0$. Since any class is equal to some $[v]$, this means that $f = 0$. The map q^* is thus injective.

(3) (3pt) Show that $\mathcal{P}$ is a sheaf of functions on $\mathbb{T}$.

(4) (2pt) Let $W \in \mathsf{Op}\,\mathbb{T}$, $f, g \in \mathcal{P}(W)$ and $\lambda, \mu \in \mathbb{R}$. Verify that $\lambda f + \mu g$ and $f \cdot g$ are in $\mathcal{P}(W)$. Conclude that $(\mathbb{T}, \mathcal{P})$ an $\mathbb{R}$-space.

Solution. Same proof as for examples of the course.

Remark. The fact that global functions on $\mathbb{T}$ (defined as elements of $\mathcal{P}(\mathbb{T})$) are periodic is very natural. In fact, one can also see that functions in $\mathcal{P}(W)$ correspond to smooth periodic functions defined on $q^{-1}(W)$ (which is naturally stable by shifts by $\mathbb{Z}^2$). The interested reader may formalise this as a separate exercise.

2. Charts for the torus. *In order to prove that the $\mathbb{R}$-space obtained above is a smooth manifold, we need to provide charts. The construction of candidates for the charts was done in the last homework.*

Let us recall that back there, we discussed $U_v = B(v, 1/3)$, $q(U_v)$ and $S_v = q^{-1}(q(U_v))$. We constructed a continuous map and $\sigma_v : S_v \to U_v$ and factored it as $\sigma_v = s_v \circ q|_{S_v}$. The continuous map $s_v : q(U_v) \to U_v$ satisfies $q|_{U_v} \circ s_v = \mathrm{id}_{q(U_v)}$ and $s_v \circ q|_{U_v} = \sigma_v|_{U_v} = \mathrm{id}_{U_v}$. It is thus a homeomorphism.

(1) (2pt) Show that $\sigma_v|_{U_{v+(n,m)}} : U_{v+(n,m)} \to U_v$ is a C^∞-diffeomorphism. Conclude that σ_v is smooth on S_v.

Solution. $\sigma_v|_{U_{v+(n,m)}}$ acts as $w \mapsto w - (n, m)$. This is an affine map (hence smooth), with inverse $w \mapsto w + (n, m)$. We then conclude since smooth functions form a sheaf and $\{U_{v+(n,m)}\}_{n,m}$ cover S_v.

(2) (1pt) Let W be an open subset of U_v. Show that

$$q^{-1}(s_v^{-1}(W)) = \sqcup_{m,n} W_{m,n}, \quad W_{m,n} = \{(x, y) \in \mathbb{R}^2 \quad | \quad (x - m, y - n) \in W\}.$$

Solution. $q^{-1}s_v^{-1}(W) = \sigma_v^{-1}(W) = \{(x, y) \in S_v | \sigma_v(x, y) \in W\}$. The latter is equal to $\sqcup_{m,n}\{(x, y) \in U_{v+(m,n)} | (x - m, y - n) \in W\}$. We can replace $U_{v+(m,n)}$ by $\mathbb{R}^2$ here: if $(x, y) \in \mathbb{R}^2$ is such that $(x-m, y-n) \in W$, then $(x-m, y-n) \in U_v$ and so $(x, y) \in U_{v+(m,n)}$, so this is not a condition.

(3) (3pt) Let $f \in C^\infty_{\mathbb{R}^2}(W)$. Show that $s_v^*(f) : s_v^{-1}(W) \to \mathbb{R}$ belongs to $\mathcal{P}(s_v^{-1}(W))$ where $\mathcal{P}$ is defined as above.

Solution. We need to apply q^* to it. But $q^*s_v^*(f) = \sigma_v^*(f)$, and since σ_v is a smooth map between opens in $\mathbb{R}^2$, precomposing a locally defined smooth functions with σ_v gives a smooth function $\sigma^*(f) : q^{-1}(s_v^{-1}(W)) \to \mathbb{R}$.

(4) (3pt) Let $f : W \to \mathbb{R}$ be any function such that $s_v^*(f) : s_v^{-1}(W) \to \mathbb{R}$ belongs to $\mathcal{P}(s_v^{-1}(W))$. Prove that f is smooth on W.

Solution. Apply q^* to it. Just like above, we get that $\sigma^*(f) : q^{-1}(s_v^{-1}(W)) \to \mathbb{R}$ is smooth. But $W = W_{0,0} \subset q^{-1}(s_v^{-1}(W))$, and so $\sigma_v^*(f)|_W : W \to \mathbb{R}$ is smooth. Finally, $\sigma_v|_W = \mathrm{id}_W$, and so $\sigma_v^*(f)|_W = f \circ \sigma_v|_W = f$.

(5) (2pt) Show that one can find a cover $\mathbb{T}$ consisting of a finite amount of sets of the form $q(U_{v_i})$ for some $v_i \in \mathbb{R}^2$. Solution. One can definitely find a cover of $\mathbb{T}$ by the infinite family $\{q(U_v)\}_{v\in\mathbb{R}^2}$. Note that $\mathbb{T} = q([0, 1] \times [0, 1])$, and the latter is compact in $\mathbb{R}^2$. Images of compacts being compacts, we can choose a finite subcover of any cover, meaning there exist $v_1, \ldots, v_n$ such that $\mathbb{T} = q(U_{v_1}) \cup \ldots \cup q(U_{v_n})$.

Of course, there are ways to produce such a cover explicitly.

This shows that $(\mathbb{T}, \mathcal{P})$ *is a smooth manifold: for each* $p \in \mathbb{T}$ *there is a set* $q(U_v)$ *containing* p *that is homeomorphic (via* $s_v : q(U_v) \xrightarrow{\sim} U_v$*)) to an open subset of* $\mathbb{R}^2$*, and this is an* $\mathbb{R}$*-space isomorphism.*

Homework problem set 3

20 pts max, the exercises offer more as usual. A couple of them might require some advanced handling of combinatorics, so I marked them with stars. Solving them is not necessary to get the maximum grade. Feel free to ask questions.

1. Acting on the projective line. *Let* $G = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \in \mathsf{GL}_2(\mathbb{R})$ *and consider the associated map*

$$G : \mathbb{R}^2 \to \mathbb{R}^2, \quad v \mapsto Gv.$$

Also recall the projective line equivalence relation on $\mathbb{R}^2 \setminus 0$, $v \sim w$ *iff* $v = \lambda w$ *for* $\lambda \neq 0$. *We denote* $q : \mathbb{R}^2 \setminus 0 \to \mathbb{RP}^1$ *the smooth quotient map. As usual, writing* $[x_0 : x_1]$ *denotes the line* $\mathsf{Span}((x_0, x_1))$.

(1) (2pt) Show that $v_1 \sim v_2$ implies $Gv_1 \sim Gv_2$. Conclude that there exists a continuous map $L_G : \mathbb{RP}^1 \to \mathbb{RP}^1$ that sends $\mathsf{Span}(v)$ to $\mathsf{Span}(Gv)$.

Solution. Any linear map, and in particular G, satisfies $G(\lambda v) = \lambda G(v)$. Becasuse of it $v_1 = \lambda v_2$ implies $G(v_1) = \lambda G(v_2)$. We also need to check that $G(v) = 0 \implies v = 0$, and this is true because $\det G = -4$. Thus we conclude that $G : \mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$ is a map that respects $\sim$.

From this, we see that the map $q \circ G : \mathbb{R}^2 \setminus 0 \to \mathbb{RP}^1$ is a smooth map that sends $v \equiv v'$ to $q(G(v)) = q(G(v'))$. It thus can be factored as $q \circ G = L_G \circ q$ with $L_G : \mathbb{RP}^1 \to \mathbb{RP}^1$ continuous. By definition $L_G(q(v)) = q(G(v))$, it remains to observe that in terms of lines, $q(v) = \mathsf{Span}(v)$.

(2) (2pt) Show that the map L_G is smooth.

Solution. We need to show that for any $f : \mathbb{RP}^1 \to \mathbb{R}$ such that q^*f smooth, we have $L_G^* f$ smooth. By definition, it means that $q^* L_G^* f$ should be smooth. The latter is equal to $(L_G \circ q)^* f = (q \circ G)^* f = G^*(q^* f)$. This is smooth since G is a smooth map.

Comment: Maybe it is worth noting that this is merely the corollary of the fact that q satisfies the smooth quotient universal property. Indeed, for any smooth $F : \mathbb{R}^2 \setminus 0 \to M$ where M is a smooth manifold, satisfying $F(v) = F(v')$ for $v \sim v'$, there exists unique $\bar{F} : \mathbb{RP}^1 \to M$ such that $F = \bar{F} \circ q$. Simply take $\bar{F}$ to be the one given by the property of q as a continuous map, and observe that $q^*(\bar{F}^* f) = F^* f$ is smooth due to the smoothness of F.

(3) (2pt) Find a matrix $S \in \mathsf{GL}_2(\mathbb{R})$ such that $L_G = L_S$ and $\det S = \pm 1$.

Solution. Observe that for $\lambda \neq 0$, $L_G = L_{\lambda G}$ simply because that rescaling does not change the lines. Because of this, we can take $\lambda = \pm 1/\sqrt{|\det G|} = \pm 1/2$. The matrix is thus

$$S = \pm \begin{pmatrix} 1/2 & 1 \\ 1/2 & -1 \end{pmatrix}.$$

Comment: the set $\{S \in \mathsf{GL}_2(\mathbb{R}) \quad | \quad \det S = \pm 1\}$ is a subgroup of $\mathsf{GL}_2(\mathbb{R})$, usually called $\mathbb{P}\,\mathsf{GL}_2(\mathbb{R})$ ($\mathbb{P}$ for projective).

(4) (3pt) Recall the charts $\varphi : [x_0 : x_1] \mapsto x_1/x_0$ (defined for lines with $x_0 \neq 0$) and $\psi : [x_0 : x_1] \mapsto x_0/x_1$ (defined for lines with $x_1 \neq 0$). Let $p = [1 : y]$.

Under what conditions $L_G(p)$ belongs to the chart defined by φ? What, then, is the formula for $\varphi(L_G(p))$? Same question for ψ and $\psi(L_G(p))$.

Solution. I will denote $U_0 = \{x_0 \neq 0\} \subset \mathbb{RP}^1$ and $U_1 = \{x_1 \neq 0\} \subset \mathbb{RP}^1$ the two charts corresponding to φ and ψ. We compute

$$L_G(p) = L_G([1 : y]) = q(G(1, y)) = q(1 + 2y, 1 - 2y) = [1 + 2y : 1 - 2y].$$

Now,

(a) To have $L_G(p) \in U_0$, we need $1+2y \neq 0$, and so $y \neq -1/2$. This means, geometrically, that that as a line, $p \neq \mathsf{Span}(-2, 1)$. The formula in this case is $\varphi(L_G(p)) = (1 - 2y)/(1 + 2y)$, the division is authorised.

(b) To have $L_G(p) \in U_1$, we need $1 - 2y \neq 0$, and so $y \neq 1/2$. This means, geometrically, that that as a line, $p \neq \mathsf{Span}(2, 1)$. The formula in this case is $\psi(L_G(p)) = (1 + 2y)/(1 - 2y)$, the division is authorised.

(5) (3pt) Let $p = [1 : y]$ and consider $V \in T_p\mathbb{RP}^1$ such that $\varphi_*V = \frac{d}{dt}|_y$ (here t denotes the coordinate in the real line – image of φ).

If $L_G(p)$ belongs to the chart φ, what is the formula for $\varphi_*(L_G)_*V$? Same question for $\psi_*(L_G)_*V$.

Solution. I honestly expected that people will do the following set of operations "according to lectures":

(a) If $L_G(p)$ belongs to chart φ, then the lectures tell me that for $w = ad/dt|_y$, to know $\varphi_*(L_G)_*w = bd/du|_{\varphi(L_G(p))}$, I can compute it as

$$b = J(\varphi \circ L_G \circ \varphi^{-1})(y) \cdot a.$$

In our case $a = 1$ and $\varphi \circ L_G \circ \varphi^{-1}(t) = (1 - 2t)/(1 + 2t)$ so its Jacobian at $y \neq -1/2$ is

$$J(\ldots)(y) = \left(\frac{1 - 2t}{1 + 2t}\right)'(y) = \frac{-2}{1 + 2y} - 2\frac{1 - 2y}{(1 + 2y)^2} = \frac{-4}{(1 + 2y)^2}.$$

Thus

$$\varphi_*(L_G)_*V = \frac{-4}{(1 + 2y)^2}\frac{d}{du}\bigg|_{\frac{1-2y}{1+2y}}.$$

(b) (modified copypasta) If $L_G(p)$ belongs to chart ψ, then the lectures tell me that for $w = ad/dt|_y$, to know $\psi_*(L_G)_*w = bd/du|_{\psi(L_G(p))}$, I can compute it as

$$b = J(\psi \circ L_G \circ \varphi^{-1})(y) \cdot a.$$

In our case $a = 1$ and $\psi \circ L_G \circ \varphi^{-1}(t) = (1+2t)/(1-2t)$ so its Jacobian at $y \neq 1/2$ is

$$J(\ldots)(y) = \left(\frac{1 + 2t}{1 - 2t}\right)'(y) = \frac{2}{1 - 2y} + 2\frac{1 + 2y}{(1 - 2y)^2} = \frac{4}{(1 - 2y)^2}.$$

Thus

$$\psi_*(L_G)_*V = \frac{4}{(1 - 2y)^2}\frac{d}{du}\bigg|_{\frac{1+2y}{1-2y}}.$$

Comment: alright, seems that this was difficult to derive. Let me explain the subtleties here.

What is the meaning of $\varphi_* V$ for $V \in T_p\mathbb{RP}^1$? Formally speaking, V is defined on all functions $C^\infty(\mathbb{RP}^1)$, But in fact it is defined on any function $f \in C^\infty(U)$ for $U \ni p$. We explore this in Proposition 3.29 in the course. The point is that for any local smooth function $f : U \to \mathbb{R}$ there exists open $p \in V \in U$, $g : \mathbb{RP}^1 \to \mathbb{R}$ such that $g|_V = f_V$, and we can define $V(f) := V(g)$. Proposition checks that this definition is independent of the function g. In words, we can extend any local function to a global function that agrees with the initial function on a small neighbourhood of p, and evaluating V on any such extension gives the same result. This allows to prove that $i : U \hookrightarrow M$ induces $i_* : T_pU \cong T_pM$.

The convention that we adapt is to actually identify T_pU with T_pM via the map i_*. This may seem unnatural but in fact happens in mathematics oftentimes. For example, we do not distinguish between $A \times (B \times C)$ and $(A \times B) \times C$ between two triple products of two sets, even though technically the first one consists of $(a, (b, c))$ and the second one of $((a, b), c)$. So the notation $\varphi_* V$ means that we consider $i : U_0 \subset \mathbb{RP}^1$, find unique W such that $i_* W = V$, and compute $\varphi_* W$. On a smooth function $f : \mathbb{R} \to \mathbb{R}$, we have $\varphi_* W(f) = W(f \circ \varphi) = V(g)$ where g is a global function that coincides with $f \circ \varphi : U_0 \to \mathbb{R}$ on a neighbourhood of p.

This identification of tangent spaces works, in part, because everything is usually compatible with restrictions. For example, if we have any open $U \subset U_0$, the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\varphi|_U"] \arrow[d, "j"'] & \varphi(U) \arrow[d, "k"] \\ U_0 \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

Because of this if we have any X such that $i_* X = W$, then $(k \circ \varphi|_U)_* X = \varphi_* W$. So if I have $V \in T_p\mathbb{RP}^1$ such that $i_* W = V$, then by $\varphi_* V$ we can also mean $(\varphi|_U)_* X$, cause in the end it does not matter: $k_*(\varphi|_U)_* X = \varphi_* W$ so they will work the same way on a smooth function on $\mathbb{R}$.

Arguably, this is why the definition using germs is the better one, it allows to not think in terms of global functions, so such a discussion is not necessary.

Now, with this in mind, let us interpret the problem that we are facing. If $y \neq -1/2$, then $L_G(p) \in U_0$. It does not mean that $L_G(U_0) \subset U_0$, rather that $L_G(U_0\ \{[2:-1]\}) \subset U_0$. The set $U_0\ \{[2:-1]\}$ is open, and the following diagram commutes:

$$\begin{tikzcd} U_0 \setminus \{[2:-1]\} \arrow[r, "\varphi"] \arrow[d, "L_G"'] & \mathbb{R} \setminus \{-1/2\} \arrow[d, "F"] \\ U_0 \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

In principle, I should write some restrictions, but this is understood in computations. The map F equals to $\varphi \circ L_G \circ \varphi^{-1}$ and $\varphi \circ L_G = F \circ \varphi$. In light of the discussion above, we can write $\varphi_* \circ (L_G)_* V = F_* \circ \varphi_* V$. So given that $\varphi_* V : f \mapsto f'(y)$, we compute

$$\varphi_* \circ (L_G)_* V(g) = \varphi_* V(g \circ F) = \frac{d}{dt}\bigg|_y g\left(\frac{1-2t}{1+2t}\right) = \frac{4}{(1+2y)^2} \frac{d}{du}\bigg|_{\frac{1-2y}{1+2y}} g(u). \tag{*}$$

We could have made the same computation by picking any open neighborhood $\mathcal{U}_{L_G(p)}$ of $L_G(p)$ that is contained in U_0, and take $\mathcal{U}_p := L_G^{-1}(\mathcal{U}_{L_G(p)}) \cap U_0$. This will be an open inside U_0 containing p. The diagram

$$\begin{tikzcd}
\mathcal{U}_p \arrow[r, "\varphi"] \arrow[d, "L_G"'] & \varphi(\mathcal{U}_p) \arrow[d, "F"] \\
\mathcal{U}_{L_G(p)} \arrow[r, "\varphi"] & \varphi(\mathcal{U}_{L_G(p)})
\end{tikzcd}$$

commutes and same arguments apply.

TLDR: If $L_G(p)$ belongs to chart φ, then $\varphi \circ L_G = F \circ \varphi$, $F = \varphi \circ L_G \circ \varphi^{-1}$ on some open neighbourhood of $\varphi(p)$, and $\varphi_*(L_G)_*V = F_*\varphi_*V$. If $L_G(p)$ belongs to chart ψ, then $\psi_*(L_G)_*V = H_*\varphi_*V$, $H = \psi \circ L_G \circ \varphi^{-1}$ on some open neighbourhood of $\varphi(p)$. In computations, you can reason as in (*) above without specifying the domain of g.

2. Matrix tensors. *For $n \geq 1$, denote $\langle n \rangle = \{1, ..., n\}$. An element of $x \in \mathbb{R}^n$ naturally defines a map $x : \langle n \rangle \to \mathbb{R}$, $i \mapsto x_i$. Similarly given $M \in \mathsf{Mat}_n(\mathbb{R})$, we have a map $\langle n \rangle^2 \to \mathbb{R}$, $(i, j) \mapsto M_{ij}$.*

A matrix k-tensor (with index range n) is a map $T : \langle n \rangle^k \to \mathbb{R}$ ($k \geq 0$). Denote $\mathsf{Ten}_n^k(\mathbb{R})$ the set of all matrix k-tensors. It is naturally a vector space (addition and $\mathbb{R}$-multiplication are defined as usual for functions to $\mathbb{R}$). We shall write $T_{i_1 \dots i_k}$ for the value of $T \in \mathsf{Ten}_n^k(\mathbb{R})$ at $(i_1, ..., i_k)$.

Denote $E^{i_1 \dots i_k} : \langle n \rangle^k \to \mathbb{R}$ the k-tensor that takes value 1 on $(i_1, .., i_k)$ and zero otherwise. It is easy to see that $\{E^{i_1 \dots i_k}\}_{i_j \in \langle n \rangle}$ is a basis of $\mathsf{Ten}_n^k(\mathbb{R})$ and so its dimension is n^k.

A matrix tensor $A \in \mathsf{Ten}_n^k(\mathbb{R})$ is antisymmetric if for all $(i_1, ..., i_k) \in \langle n \rangle^k$, and any k-permutation σ, one has

$$A_{i_1 \dots i_k} = (-1)^\sigma A_{i_{\sigma(1)} \dots i_{\sigma(k)}}.$$

Here $(-1)^\sigma$ is the sign of the permutation: -1 if σ can be expressed as an odd number of transpositions, and $+1$ otherwise. Denote $\Lambda_n^k(\mathbb{R}) \subset \mathsf{Ten}_n^k(\mathbb{R})$ the set of all antisymmetric k-tensors. It is a vector subspace.

(1) (2pt) Let $T \in \mathsf{Ten}_n^k(\mathbb{R})$ and $P \in \mathsf{Ten}_n^m(\mathbb{R})$. Define

$$T \otimes P : \langle n \rangle^{k+m} \to \mathbb{R}, \quad (T \otimes P)_{i_1 \dots i_k i_{k+1} \dots i_{k+m}} = T_{i_1 \dots i_k} P_{i_{k+1} \dots i_{k+m}}.$$

Let $Q \in \mathsf{Ten}_n^l(\mathbb{R})$. Show that

$$(T \otimes P) \otimes Q = T \otimes (P \otimes Q).$$

Is it always true that $T \otimes P = P \otimes T$? Justify your answer.

(2) (1pt) Show that $E^{i_1 \dots i_m} = E^{i_1} \otimes ... \otimes E^{i_m}$.

(3) (2pt) For $T \in \mathsf{Ten}_n^k(\mathbb{R})$, write

$$Alt(T)_{i_1 \dots i_k} = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^\sigma T_{i_{\sigma(1)} \dots i_{\sigma(k)}},$$

with Σ_k denoting the set of all k-permutations. Prove that Alt defines a linear map from $\mathsf{Ten}_n^k(\mathbb{R})$ to $\Lambda_n^k(\mathbb{R})$. What happens if we apply Alt to $A \in \Lambda_n^k(\mathbb{R})$?

(4) (3pt) For $A \in \Lambda_n^k(\mathbb{R})$ and $B \in \Lambda_n^m(\mathbb{R})$, define $A \wedge B := \frac{(k+m)!}{k!m!} Alt(A \otimes B)$. Prove that for $L^1, L^2, L^3 \in \Lambda_n^1(\mathbb{R}) = \mathsf{Ten}_n^1(\mathbb{R})$, one has

$$(L^1 \wedge L^2) \wedge L^3 = L^1 \wedge (L^2 \wedge L^3)$$

in $\Lambda_n^3(\mathbb{R})$.

(5**) (4pt) Show that for $A \in \Lambda_n^k(\mathbb{R})$, $B \in \Lambda_n^m(\mathbb{R})$ and $C \in \Lambda_n^l(\mathbb{R})$, one has

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

in $\Lambda_n^{k+m+l}(\mathbb{R})$ (solving this automatically solves the previous point).

(6*) (4pt) Admitting the previous point if necessary, show that a basis of $\Lambda_n^k(\mathbb{R})$ for $k \geq 1$ is given by

$$\{E^{i_1} \wedge \ldots \wedge E^{i_k}\}_{1 \leq i_1 < i_2 \ldots < i_k \leq n}$$

(proving for $k \leq 3$ will give 2pt)

Homework problem set 4: Lie groups and Lie algebras

To understand this homework, you need to read: section 4 of the course (around the statement about submanifolds as level sets, Theorem 4.17), section 5 (everything, more or less).

The following reference is strongly advised: [1] J. Lee, introduction to smooth manifolds, Chapters 7-8, an online version can be found at `https://math.berkeley.edu/~jchaidez/materials/reu/lee_smooth_manifolds.pdf`*. I am not denying that many questions are covered there, you can view this homework as an exercise of working with the literature!*

NB: Lee [1] often switches between notation $F_* \equiv F_*(p) : T_pM \to T_{F(p)}N$ and dF_p to mean the same operation of pushforward: $dF_p(V)(f) = V(f \circ F)$.

Definition 1. A *Lie group* is a smooth manifold G that also has a group structure (as a set), subject to the condition that the maps $m : G \times G \to G, (g, h) \mapsto g \cdot h$ and $\iota : G \to G, g \mapsto g^{-1}$ are smooth. A *Lie group homomorphism,* or *map*, is a smooth map $F : G \to H$ that is also a group homomorphism.

You may want to consult the beginning of PS3 to remind yourself about products of smooth manifolds.

1. Examples.

(1) Show that $\mathsf{GL}_n(\mathbb{R})$ and $\mathsf{GL}_n(\mathbb{C})$ are Lie groups. What are their manifold dimensions?

(2) Let G be a Lie group and H its subgroup (in ordinary sense). Assume that H is a smooth submanifold. Show that H is a Lie group.
Indication. See [1, Proposition 7.11] as well as Theorem 4.15 of our course. What Lee calls embedded submanifolds we may have simply called submanifolds at some point.

(3) Using this result, show that the examples below are Lie groups and compute their (manifold) dimension.
  (a) The special linear group $\mathsf{SL}_n(\mathbb{R}) = \{A \in \mathsf{GL}_n(\mathbb{R}) | \det A = 1\}$,
  (b) The orthogonal group $\mathsf{O}_n(\mathbb{R}) = \{A \in \mathsf{GL}_n(\mathbb{R}) | {}^tA \cdot A = I_n\}$,
  (c) The symplectic group $\mathsf{Sp}_{2n}(\mathbb{R}) = \{A \in \mathsf{GL}_{2n}(\mathbb{R}) | {}^tAJ_{2n}A = J_{2n}\}$ where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$,
Indication. The first two questions were already treated in TD, so you are invited to see how to do it, and reproduce. The key is to use Theorem 4.17. The book [1] is also of help. I accept Jacobi formula without proof. The symplectic group example is interesting but not easy, you can try to look it up on stackexchange, or do small $n = 1, 2$ cases that I will also accept, or think about reasoning like `https://tinyurl.com/yx3smk5z`.

(4) In PS4 we encountered the quaternion algebra $\mathbb{H} = \{x^0\mathbf{1} + x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}\}$ that we identify with $\mathbb{R}^4 = \{(x^0, x^1, x^2, x^3)\}$.
  (a) Write explicit formulas for the multiplication map $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$, $(q, q') \mapsto q \cdot q'$ and the inversion map $\mathbb{H} \setminus 0 \to \mathbb{H} \setminus 0$, $q \mapsto q^{-1}$. Conclude that these maps are smooth.
  (b) Show that $\mathbb{H}^* = \mathbb{H} \setminus 0$ is a Lie group with respect to multiplication.
  (c) Find a Lie (sub)group structure on $\mathbb{S}^3$.
Indication. Look up the PS4. The sphere $\mathbb{S}^3$ sits inside $\mathbb{H}^*$ as a certain kind of quaternions and you have to check that they form a subgroup. We don't distinguish between $\mathbb{H}$ and $\mathbb{R}^4$ in terms of smooth structure, so you do not have to reprove anew that $\mathbb{S}^3 \subset \mathbb{H}^*$ is a smooth embedded submanifold etc.

2. Lie algebra of left-invariant vector fields. The goal of this exercise is to show that the tangent space to the unit element of a Lie group carries a bracket satisfying the Lie algebra axioms (hence the name). This is done by identifying a certain class of vector fields on G that are canonically determined by their value at the unit $e \in G$.

Let G be a Lie group. Note that for each $g \in G$, the map

$$L_g = m(g, -) : G \to G, \quad h \mapsto L_g(h) = gh$$

is smooth: it is a composition of $G \to G \times G, h \mapsto (g, h)$ which is smooth, and the multiplication map. It has a smooth inverse given by $L_{g^{-1}}$.

Definition 2. A vector field $X \in \mathcal{T}_G(G)$ is *left-invariant* if for each $g \in G$, we have $(L_g)_* X = X$ (here $(L_g)_*$ means the pushforward by the diffeomorphism L_g as in Proposition 5.20). Equivalently, for each $g, h \in G$ and $f \in C^\infty(G)$, we have $X_h(f(g-)) \equiv (L_g)_* X_h(f) = X_{gh}(f)$, $f(g-) = f \circ L_g$.

We denote $\mathcal{L}(G)$ the set of all left-invariant vector fields on G.
Indication. For this whole exercise, [1, Theorem 8.37] is extremely useful. We basically reproduce their exposition, so I suggest to look things up.

(1) Show that $\mathcal{L}(G)$ is stable in $\mathcal{T}_G(G)$ by linear sums and the Lie bracket: in other words, that it is a Lie subalgebra.
Indication. Check out [1, Proposition 8.33] and Proposition 5.20 of our course.
(2) Let $X \in \mathcal{L}(G)$ be a left invariant vector field. Show that its values $X_g \in T_g G$ are uniquely determined by X_e and L_g, where $e \in G$ is the unit of the group. Use this to show that the (linear) map
$$ev_e : \mathcal{L}(G) \to T_e G, \quad X \mapsto X_e$$
is injective.
(3) Consider a smooth curve $\gamma : I =]-\epsilon, \epsilon[\to G$ such that $\gamma(0) = e$. Let $V = \gamma'(t) = \gamma_*(d/dt|_0)$. We consider now the family $V^L = \{V^L_g = (L_g)_* V\}_{g \in G}$. The goal is to show that it is a smooth left-invariant vector field.
  (a) For $f \in C^\infty(G)$, show that $V^L_g(f) = \frac{d}{dt}\big|_{t=0}(f \circ L_g \circ \gamma)$.
  (b) Denote
  $$\Gamma(t, g) := (f \circ L_g \circ \gamma)(t) = f(g \cdot \gamma(t)) = f(m(g, \gamma(t))).$$
  Why is $\Gamma : I \times G \to \mathbb{R}$ smooth? Show that the assignment
  $$g \mapsto \frac{d}{dt}\bigg|_{t=0} \Gamma(t, g) = \lim_{t \to 0} \frac{\Gamma(t, g) - \Gamma(0, g)}{t}$$
  is smooth as a map $G \to \mathbb{R}$.
  Indication. Once you have explained the smoothness of Γ, it may be useful to take a chart $\varphi : U \cong \Omega$ of G, and first explain why
  $$p \in \Omega \mapsto \lim_{t \to 0} \frac{\Gamma(t, \varphi^{-1}(p)) - \Gamma(0, \varphi^{-1}(p))}{t}$$
  is a smooth map $\Omega \to \mathbb{R}$.
  (c) Use this to conclude that $V = \{V^L_g\}$ is a smooth left-invariant vector field.
  Indication. Don't forget to check the left-invariance!
(4) Conclude that the (linear) map
$$ev_e : \mathcal{L}(G) \to T_e G, \quad X \mapsto X_e$$
is surjective. Indication. Lemma 3.37 of our course.

3. Induced homomorphisms. Before we move to examples, it is useful to understand how Lie group maps interact with the construction $G \mapsto \mathcal{L}(G)$. Indeed, one basic example of a Lie group map is the subgroup inclusion $H \hookrightarrow G$.

(1) We first address the preservation of Lie brackets by smooth maps. Let $F : M \to N$ be a smooth map. Let X_1, X_2 be two vector fields on M and Y_1, Y_2 be two vector fields on N. Assume further that X_i is F-related to Y_i (Definition 5.18). We want to show that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.
   (a) First, prove that $X \in \mathcal{T}_M(M)$ is F-related to $Y \in \mathcal{T}_N(N)$ iff for all $f \in C^\infty(N)$, one has $X(f \circ F) = Y(f) \circ F$, an equality of functions on M.
   Indication. What does it mean at each point $p \in M$? Suggesting [1, Proposition 8.16].
   (b) Second, use this to show that in the situation above, $[X_1, X_2](f \circ F)$ coincides with $[Y_1, Y_2](f) \circ F$.
   Indication. Suggesting [1, Proposition 8.30].
   This concludes the proof of F-relation of Lie brackets.
(2) We now use it in Lie group context. Let $F : G \to H$ be a Lie group homomorphism. Let $X \in \mathcal{L}(G)$ be a left-invariant vector field on G. Show that there exists unique left invariant vector field $Y \in \mathcal{L}(G)$ that is F-related to X.
   Indication. You may want to take $Y = (F_*(e)X_e)^L$ with $F_*(e) : T_eG \to T_eH$ the standard pushforward map, that Lee also calls dF_e. See [1, Theorem 8.44].
(3) Conclude that for each Lie group homomorphism $F : G \to H$ there is an induced Lie algebra homomorphism (Definition 5.16) $\mathcal{L}(F) : \mathcal{L}(G) \to \mathcal{L}(H)$ that assigns to $X \in \mathcal{L}(G)$ the unique F-related $Y \in \mathcal{L}(H)$. Show that $\mathcal{L}(\mathrm{id}_G)$ is an identity and $\mathcal{L}(F_2 \circ F_1) = \mathcal{L}(F_2) \circ \mathcal{L}(F_1)$.
   Indication. To verify the properties of the last sentence, observe that it is enough to check what happens at the tangent space(s) of the unit(s).
(4) let H be a lie subgroup of G and denote $i : H \to G$ the inclusion map (a Lie group homomorphism). Show that $\mathcal{L}(i)(\mathcal{L}(H))$ is a Lie subalgebra of $\mathcal{L}(G)$ and that
$$\mathcal{L}(i)(\mathcal{L}(H)) = \{X \in \mathcal{L}(G) | X_e \in i_*(T_eH)\}.$$
   Indication. Try to see more generally why given a Lie algebra homomorphism $f : \mathfrak{h} \to \mathfrak{g}$, the image $f(\mathfrak{h})$ is stable under Lie brackets of $\mathfrak{g}$. Also see [1, Theorem 8.46].

We have shown the validity of the following:

Theorem 1. *The isomorphism $ev_e : \mathcal{L}(G) \cong T_eG =: \mathfrak{g}$ induces a Lie algebra structure on $\mathfrak{g}$. Given $V, W \in \mathfrak{g}$, the bracket is given by*
$$[V, W]_{\mathfrak{g}} := [V^L, W^L]_e = ev_e([V^L, W^L]) = ev_e([ev_e^{-1}(V), ev_e^{-1}(W)]).$$
Furthermore, any Lie group map $F : H \to G$ induces a homomorphism of Lie algebras $F_* : \mathfrak{h} = T_eH \overset{F_*(e)}{\to} T_eG = \mathfrak{g}$. In particular, if $i : H \hookrightarrow G$ a Lie subrgoup inclusion, then $i_*\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

4. Geometric examples of Lie algebras. We can use the acquired knowledge to do some examples of Lie algebras.

(1) Consider again $\mathsf{GL}_n(\mathbb{R}) = \{X = (x^{ij}) | \det X \neq 0\}$, and denote $\mathfrak{g} := T_{I_n}\mathsf{GL}_n(\mathbb{R})$. Given any $V \in \mathfrak{g}$ we can write it as $V = \sum A^{ij} \frac{\partial}{\partial x^{ij}}\big|_{I_n}$, providing a natural isomorphism $\mathfrak{g} \cong \mathsf{Mat}_n(\mathbb{R})$.
   Let $V = \sum A^{ij} \frac{\partial}{\partial x^{ij}}\big|_{I_n}$ and $W = \sum B^{ij} \frac{\partial}{\partial x^{ij}}\big|_{I_n}$ be two vectors of $\mathfrak{g}$. Show that
$$[V, W]_{\mathfrak{g}} = \sum [A, B]^{ij} \frac{\partial}{\partial x^{ij}}\bigg|_{I_n}$$
   where $[A, B]^{ij}$ denotes the coefficients of the matrix commutator between A and B. This means that $\mathfrak{g} \cong \mathfrak{gl}_n(\mathbb{R})$, where the latter is $\mathsf{Mat}_n(\mathbb{R})$ as a vector space with commutator as the Lie bracket.

Indication. For $X \in \mathsf{GL}_n(\mathbb{R})$, you need to check that $V^L_X(f) = V(f \circ L_X) = \sum (XA)^{ij}\partial_{ij} f(X)$. The rest is then a direct computation, aided by [1, Proposition 8.41] (remember that Lee sums implicitly over all repeating indices).

(2) For each Lie subgroup $i : H \hookrightarrow \mathsf{GL}_n(\mathbb{R})$, we know from the theorem above that $i_*(T_{I_n}H)$ is a Lie subalgebra of $\mathfrak{g}$. This implies that its image in $\mathfrak{gl}_n(\mathbb{R})$ is stable under commutators!

In each example below, find $i_*(T_{I_n}H)$, its image under the isomorphism $\mathfrak{g} \cong \mathfrak{gl}_n(\mathbb{R})$, and verify explicitly that the resulting subset of matrices in $\mathfrak{gl}_n(\mathbb{R})$ is stable under matrix commutation.

(a) $H = \mathsf{SL}_n(\mathbb{R})$,
(b) $H = \mathsf{O}_n(\mathbb{R})$,
(c) $H = \mathsf{Sp}_{2n}(\mathbb{R})$.

Indication. You may want to consult the end of solutions to TD3, [1, Example 8.47] and maybe flesh out arguments like `https://tinyurl.com/p6mfpmjd` (here, you can assume that you have proven that $\mathsf{Sp}_{2n}(\mathbb{R})$ is a Lie subgroup of $\mathsf{GL}_{2n}(\mathbb{R})$, even if you have not done the corresponding point above).

(3) We finally get back to quaternions $\mathbb{H}$ and $i : \mathbb{S}^3 \hookrightarrow \mathbb{H}^* = \mathbb{H} \setminus 0$. We would like to understand more about the Lie algebra of $\mathbb{S}^3$.

(a) Let $q \in \mathbb{H}^*$ and Q be a vector field on $\mathbb{H}^*$ defined by

$$Q_x = \sum_\alpha (x \cdot q)^\alpha \frac{\partial}{\partial x^\alpha}\bigg|_x$$

Verify that Q is left-invariant: for each $r \in \mathbb{H}^*$ and $f : \mathbb{H}^* \to \mathbb{R}$ smooth, one has $Q_x(f \circ L_r) = Q_{r \cdot x}(f)$.

Indication. Explicit computation.

(b) Take $q = \mathbf{i}, \mathbf{j}, \mathbf{k}$ and verify that the resulting vector fields I, J, K given by

$$I_x = \sum_\alpha (x \cdot \mathbf{i})^\alpha \frac{\partial}{\partial x^\alpha}\bigg|_x, \quad J_x = \sum_\alpha (x \cdot \mathbf{j})^\alpha \frac{\partial}{\partial x^\alpha}\bigg|_x, \quad K_x = \sum_\alpha (x \cdot \mathbf{k})^\alpha \frac{\partial}{\partial x^\alpha}\bigg|_x$$

are linearly independent at each $x \in \mathbb{H}^*$ and are tangent to $\mathbb{S}^3$ for each $x \in \mathbb{S}^3$, meaning that $I_x, J_x, K_x \in i_*(T_x\mathbb{S}^3)$.

Indication. This should look extremely familiar to PS4, except that the order of multiplication is different! The reasoning is the same, the multiplication order changed since we are dealing with left invariance here.

(c) Let X be a smooth vector field on $\mathbb{H}^*$ such that $X_x \in i_*(T_x\mathbb{S}^3)$ for each $x \in \mathbb{S}^3$. We know from Problem 1 of PS4 that there exists unique vector field $Y \in \mathcal{T}_{\mathbb{S}^3}(\mathbb{S}^3)$ such that $i_*(x)Y_x = X_x$. Show that if X is left-invariant on $\mathbb{H}^*$, then Y is left-invariant on $\mathbb{S}^3$.

Indication. The inclusion $i : \mathbb{S}^3 \hookrightarrow \mathbb{H}^*$ is a Lie group homomorphism, so you may have derived already that $i \circ L_q = L_q \circ i$ for $q \in \mathbb{S}^3$. If not, derive it.
Then, to show that $(L_q)_* Y_x = Y_{qx}$ it is enough, using the injectivity of i_*, to show that $i_*(L_q)_* Y_x = i_* Y_{qx}$. Use the previously derived equality to exchange the order of maps on the left hand side, and complete the reasoning.

(d) The preceding result implies that the fields I, J, K can be restricted to $\mathbb{S}^3$ giving an explicit basis of $\mathcal{L}(\mathbb{S}^3)$. This also means that $\mathsf{Span}(I, J, K)$ is a Lie subalgebra of $\mathcal{L}(\mathbb{H}^*)$, or of $\mathcal{T}(\mathbb{H}^*)$.
Verify that claim by explicitly computing $[I, J]$, $[J, K]$, $[K, I]$ (and finding unsurprising answers).

Questions on course material

Two of these questions will appear in the test of 19/05 and will be worth, in totality, 12 points. Some statements, like question 2 below may have been skipped in the course. It is your work to find a proof or ask a question to your instructors if you do not see how to answer a question.

(1) What is a topological space? A continuous map? A homeomorphism? Give an example of a homeomorphic and a non-homeomorphic pair of spaces, with an explanation.

Indication. Lecture 1. You can also use Problem set 2, exercise one for an example of homeomorphic spaces ($\mathbb{R}^n$ and disk). For non-homeomorphic, you could take an extreme example of a finite set (a point) with discrete topology and something infinite ($\mathbb{R}$) and reason that any homeomorphism is in particular a bijection.

(2) Let X be a topological space and S a subset. What does it mean for S to be compact (in induced topology)? To be Hausdorff? Show that a compact subset of a Hausdorff space is closed.

Indication. See Problem set 1 for definition of compact, Definition 1.16 and Remark 3.6 p. 57 of the updated slides ("all in one file").

(3) What is a topological manifold? Why is $\mathbb{R}^n$ a topological manifold? Are there other examples besides opens in $\mathbb{R}^n$? Name one and sketch a proof.

Indication. Long question, eh. It is not hard to answer but requires some mental preparation. That's Definition 1.18. For a non-affine example as we say, I would probably explain $\mathbb{S}^1$ (no need to work with $\mathbb{S}^n$), and that is a simplified version of Example 1.20. In principle, you could also do $\mathsf{SL}_2(\mathbb{R})$ as in Problem Set 3, or $\mathbb{RP}^1$. I will not complain too much if you don't prove Hausdorff properties here.

Here is my answer for this question (since it requires some proofs): a topological manifold is a Hausdorff topological space M such that for each point $x \in M$ there exists an open neighbourhood $U \ni x$ and a homeomorphism $\varphi : U \xrightarrow{\sim} \Omega$ where Ω is some open in $\mathbb{R}^n$. The space $\mathbb{R}^n$ is Hausdorff as sufficiently small balls will not intersect, and we can take $U = \mathbb{R}^n$ and $\varphi = \mathsf{id}_{\mathbb{R}^n}$, so $\mathbb{R}^n$ is a topological manifold.

There are other examples. For instance, we can take $\mathbb{S}^1 \subset \mathbb{R}^2$ defined as $x^2 + y^2 = 1$. We put induced topology on it. Since $\mathbb{R}^2$ is Hausdorff, we can take two non-intersecting open balls around $p, q \in \mathbb{S}^1$ and intersect them with $\mathbb{S}^1$. The resulting open sets of $\mathbb{S}^1$ show that the latter is Hausdorff.

Next, let us consider a subset $U_x^+ := \mathbb{S}^1 \cap \{(x, y) | x > 0\}$. This is open in $\mathbb{S}^1$ by definition. Take a map $\varphi_x : U_x^+ \to]-1, 1[$ that sends (x, y) to y. It is continuous as a restriction of continuous map $(x, y) \mapsto y$ to $\mathbb{S}^1$. Its image is indeed in $]-1, 1[$ since $x^2 + y^2 = 1$ and $x > 0$ imply $y^2 < 1$.

There is also a map $\eta_x^+ : t \mapsto (\sqrt{1 - t^2}, t)$. It is defined on $]-1, 1[$, and as a map to $\mathbb{R}^2$ it is continuous and has its image land entirely in $U_x^+ \subset \mathbb{S}^1$. By a Lemma of the course (Lemma 1.22 of full notes but you can be vague here) this means that $\eta_x^+ :]-1, 1[\to U_x^+$ is continuous, and we verify that it is inverse to φ_x.

We can similarly consider $U_x^- = \{(x,y) \in \mathbb{S}^1 | x < 0\}, U_y^+ = \{(x,y) \in \mathbb{S}^1 | y > 0\}, U_y^- = \{(x,y) \in \mathbb{S}^1 | y < 0\}$ where we again use φ_x on U_x^- and $\varphi_y : (x,y) \mapsto x$. So all these sets are homeomorphic to $]-1,1[$ in $\mathbb{R}$. Since any $p \in \mathbb{S}^1$ has one of its coordinates positive or negative, it belongs to some $U_x^+, U_x^-, U_y^+, U_y^-$, and that completes the proof.

(4) What is a presheaf of functions? When is it a sheaf? Give an example of a presheaf that is not a sheaf, and of a presheaf that is a sheaf, with explanations.

Indication. Not much to say here. Just re-read the chapter on sheaves and look into PS2.

(5) What is a smooth (C^∞) manifold? Why is $\mathbb{R}^n$ a smooth manifold? Are there other examples besides opens in $\mathbb{R}^n$? Name one and present its smooth manifold structure (without proof).

Indication. So the answer should be of the sort $(M, \mathcal{A})$ where you define the topology on M, the value, $\mathcal{A}(U)$ on an open U, and a collection of maps $\varphi_i : U_i \to \Omega_i$ that cover M. Note that now you can use your torus homework. $M = \mathbb{T}$ with quotient, functions are defined to be smooth if their pullbacks by $q : \mathbb{R}^2 \to \mathbb{T}$ are such, and open charts given by $s_v : q(U_v) \to U_v$ where you recall the definition of U_v and s_v from the homework (all the proofs are done there, not here!).

(6) What is a smooth map of smooth manifolds? Name one valid criterion as explained in the course. Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth in ordinary sense iff it is smooth as a map of manifolds.

Indication. A smooth map is a continuous map such that Proposition 3.14. Then you do exercise 6 of PS2 (has correction on Moodle).

(7) What is a smooth map of smooth manifolds? Name one valid criterion as explained in the course. Give an example of a smooth map that is not a smooth map between opens in Euclidean spaces (without proof). What happens if we compose smooth maps? Justify your response.

Indication. A smooth map is a continuous map such that Proposition 3.14. Well you could recall the map $L_G : \mathbb{RP}^1 \to \mathbb{RP}^1$ from your homework maybe? Then you either do PS2 exercise 5(a) or use the global function criterion to simply explain that repeated pullback of a global smooth function is smooth.

(8) Tangent space to a manifold, how is it defined? What happens in the case $M = \mathbb{R}^n$, can we find an explicit basis? Outline the proof of that statement (Taylor formula can be admitted).

Indication. The definition is via derivations. What is asked next is to show that all derivations look like $\sum v^i \partial_i|_p$. This is Proposition 3.23, and you admit that any smooth function f on $\mathbb{R}^n$ can be decomposed as $f(x) = f(p) - \sum_i (x^i - p^i) g_i(x)$ for g_i smooth and $p = (p^1, ..., p^n)$. You will have to mention why a p-derivation X satisfies $X(f) = 0$ if f is constant (thankfully the proof is easy, Lemma 3.21(2)).

(9) Tangent space to a manifold, how is it defined? What is the relation between T_pU and T_pM for $U \subset M$ open (no proof)? What does it imply about the dimension of T_pM?

Indication. Proposition 3.29. Strictly speaking you may want to describe how i_* works, and that is 3.26. Then to cut angles you could say that any point $x \in M$ admits a chart diffeomorphic to $B(0, \varepsilon)$ or $\mathbb{R}^n$ and use Proposition 3.23 to conclude that the dimension is n.

(10) Given a smooth map $F : M \to N$, how do we define its generalised differential/pushforward $F_* : T_pM \to T_{F(p)}N$? Why is it a linear map? Given smooth maps G, F how does one compute $(G \circ F)_*$ (with proof)?

Indication. Definition 3.26 and Proposition 3.28.

(11) Given a smooth map $F : M \to N$, how do we define its generalised differential/pushforward $F_* : T_pM \to T_{F(p)}N$? If we consider smooth $F : \mathbb{R}^n \to \mathbb{R}^m$, what is the relation between F_* and $dF(p)$? Provide a justification.

Indication. Definition 3.26 and Example 3.27.

(12) Admitting how tangent spaces and pushforwards work, relate T_pS^n to a subspace in $T_p\mathbb{R}^{n+1}$, justifying your steps.

Indication. Example 3.31. You may also want to see how the tangent space to $\mathsf{SL}_2(\mathbb{R})$ was done in PS3.

(13) When is a smooth map a submersion? An immersion? Give example of each. State the canonical submersion/immersion theorem (without proof). Indication. Definition 4.1, and Theorem 4.7. Be lazy and provide π, ι as your examples of submersion and immersion.

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Duration: 1 hour. No materials are allowed. As usual, you can use an exercise to solve another exercise, even if you do not know how to do the former. Ask questions if you do not understand the formulations!

(1) (6pt)(question 8) Tangent space to a manifold, how is it defined? What happens in the case $M = \mathbb{R}^n$, can we find an explicit basis? Outline the proof of that statement (Taylor formula can be admitted).

Solution. We define $T_pM = \{D : C^\infty(M) \to \mathbb{R} | D(fg) = f(p)D(g) + D(f)g(p)\}$. When $M = \mathbb{R}^n$, then we can explicitly describe it as follows. Any function $f \in C^\infty(\mathbb{R}^n)$ can be presented as $f(x) = f(p) + \sum_i (x^i - p^i)g_i(x)$ for g_i smooth, this is the statement of the first order Taylor formula (with remainder). It immediately implies that $\partial_i f(p) = g_i(p)$.

For the moment we agree that $X(c) = 0$ for constant functions c. Then $X(f) = X(f(p) + \sum_i (x^i - p^i)g_i) = \sum_i (X(x^i - p^i)g_i(p) + (p^i - p^i)X(g_i)) = \sum_i X(x_i)\partial_i f(p)$. For constant functions, we have $X(c) = cX(1) = cX(1 \cdot 1) = 2cX(1)$, so it can be only equal to 0.

Thus $T_p\mathbb{R}^n$ is spanned by $\partial_1|_p, ..., \partial_n|_p$. To show that it is a basis, it is sufficent to evaluate any linear combination of partial derivatives at all coordinate functions x^i.

(2) (6pt) Pick your favourite question (except 8) from the list on the back and answer it!

(3) Let us consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x^2 - y^2, x^2)$.

(a) (1.5pt) Solve $F(x, y) = 0$. Show that F respects the equivalence relation on $\mathbb{R}^2 \setminus 0$ that defines the quotient $q : \mathbb{R}^2 \setminus 0 \to \mathbb{RP}^1$.

Solution. $F(x, y) = 0$ implies $x = y = 0$. If $(x, y) = \lambda(a, b)$, then $(x^2 - y^2, x^2) = (\lambda^2 a^2 - \lambda^2 b^2, \lambda^2 a^2) = \lambda^2(a^2 - b^2, a^2)$. So $F(x, y) \sim F(a, b)$.

(b) (2pt) Show that F induces a smooth map $\bar{F} : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that $\bar{F} \circ q = q \circ F|_{\mathbb{R}^2 \setminus 0}$.

Solution. The first question assures that F defines a map from $\mathbb{R}^2 \setminus 0$ to itself. We can take the composition $q \circ F$, according to the same question it sends equivalent points to equal classes. It hence can be factored as $qF = \bar{F}q$, where $\bar{F} : \mathbb{RP}^1 \to \mathbb{RP}^1$ is continuous. It is furthermore smooth: for each $f : \mathbb{RP}^1 \to \mathbb{R}$ smooth, we have $q^*(\bar{F}^* f) = (\bar{F}q)^* f = F^* q^* f$, $q^* f$ is smooth by definition, and F is a smooth map (written using quadratic functions).

(c) (1.5pt) We recall the standard chart $\varphi[x : y] = y/x$ defined for lines $[x : y]$ with $x \neq 0$. Let $p = [1 : y] \in \mathbb{RP}^1$.

When is it possible to write $\varphi(\bar{F}(p))$? What is it equal to, as a function of y, in these cases?

Solution. We have $\bar{F}([x : y]) = [x^2 - y^2 : x^2]$. For it to belong to chart φ, we need to have $x^2 - y^2 \neq 0$. In other words, p should not be one of the lines $y = \pm x$. Since our point p satisfies $\varphi(p) = y$, it means that $y \neq \pm 1$. If that is the case, we can write $\varphi(\bar{F}(p)) = 1/(1 - y^2)$.

(d) (2pt) Let $V \in T_p\mathbb{RP}^1$ such that $\varphi_* V = \frac{d}{dt}|_y$. Compute $\varphi_* \bar{F}_* V$ when it is possible. The answer should be presented as

$$\varphi_* \bar{F}_* V = A(y) \frac{d}{dt}\bigg|_{\varphi(\bar{F}(p))},$$

where you are to compute $A(y)$ and $\varphi(\bar{F}(p))$ in terms of y.

Solution. We fix $y \neq \pm 1$. We have that the Jacobian of the function $t \mapsto 1/(1-t^2)$ at y is

$$J(y) = \frac{2y}{(1-y^2)^2}$$

and so, according to the lecture notes/HW3,

$$\varphi_* \bar{F}_* V = \frac{2y}{(1-y^2)^2} \frac{d}{dt}\bigg|_{1/(1-y^2)}.$$

(4) Consider $X = \mathbb{R}$. For each example above, verify whether it is a presheaf and a sheaf. You can admit that continuous functions form a sheaf.

(a) (1pt) We set $\mathcal{A}(U) = \{f : U \to \mathbb{R} \quad | \quad f(x) \geq 0 \quad \forall x \in U\}$.

Solution. It is a sheaf. The checks are trivial to be honest, in fact we can simply observe that $\mathcal{A}(U) = \mathcal{F}(U, \mathbb{R}_{\geq 0})$. It is here for "if nothing else".

(b) (2pt) We set

(i) $\mathcal{P}(U) = C(U, \mathbb{R})$ if $U \subset \mathbb{R}$ is bounded,

(ii) $\mathcal{P}(U) = \{f : U \to \mathbb{R} | f = 0\}$ otherwise.

Solution. It is a presheaf. To see this, consider two cases.

(i) U is bounded. Then any subset V of it is also bounded. We use then the fact that a restriction of a continuous function is continuous.

(ii) U is unbounded. Then we are dealing with $f(t) = 0$ on U. Then either $V \subset U$ is also unbounded, in which case we can restrict the zero function to the zero function, or V is bounded, and so $\mathcal{P}(V) = C(V, \mathbb{R})$, but this set still contains the zero function.

It is not a sheaf. It is enough to produce a cover of $\mathbb{R}$ by bounded intervals, and consider the restriction of $\mathrm{id}_{\mathbb{R}} : t \mapsto t$ to each of these sets. Thus $\mathrm{id}_{\mathbb{R}} \in \mathcal{P}(\mathbb{R}) = \{0\}$, impossible.

(c) (2pt) We set

(i) $\mathcal{T}(U) = C(U, \mathbb{R})$ if $U \cap [-1, 1]$ is nonempty,

(ii) $\mathcal{T}(U) = \{f : U \to \mathbb{R} | f = 0\}$ otherwise.

Solution. Consider $U = \mathbb{R}$ and $V =]-2, 2[$. We have that $V \subset U$, $U \cap [-1, 1] = [-1, 1]$ and $V \cap [-1, 1] = \emptyset$. We then have $\mathrm{id}_{\mathbb{R}} \in \mathcal{T}(U)$ but its restriction cannot belong to $\mathcal{T}(V) = \{0\}$. $\mathcal{T}$ is not even a presheaf.

(d) (2pt) We are given a surjective continuous map $\pi : X \to \mathbb{R}$ and we set

$$\mathcal{S}(U) = \{f \in C(U, X) | \pi \circ f = \mathrm{id}_U\}.$$

Solution. If $f : U \to X$ satisfies $\pi f = \mathsf{id}_U$, this means that for each $x \in U$, we have $\pi(f(x)) = x$. So this is true for any $x \in V \subset U$. A restriction of a continuous function is continuous, so $\mathcal{S}$ is thus a presheaf.

If we have $U = \cup_i U_i$ and $f : U \to X$ is such that $f|_{U_i} \in \mathcal{S}(U_i)$, then

(i) f is continuous on U since continuous functions form a presheaf.

(ii) for each $x \in U$, we have $x \in U_i$ for some i. Thus we have $\pi(f(x)) = \pi(f|_{U_i}(x)) = x$.

Thus $\mathcal{S}$ is a sheaf.

Test May 31 2021

Duration: 1 hour. No materials are allowed. As usual, you can use an exercise to solve another exercise, even if you do not know how to do the former. Ask questions if you do not understand the formulations!

(1) (6pt) Answer a theory question given to you by the instructor, or choose your own and lose 1.5 pts.

(2) (6pt) Answer a theory question given to you by the instructor, or choose your own and lose 2 pts.

(3) We consider the smooth map $u : \mathbb{R}^2 \setminus 0 \to \mathbb{S}^1$, $u(v) = v/\|v\|$.

   (a) (1.5pt) Let $p = (x, y)$ with $x > 0$. Find a chart $\varphi : U \to \Omega \subset \mathbb{R}$ of $\mathbb{S}^1$ such that $u(p) \in U$.

   (b) (3pt) Let $V \in T_p\mathbb{R}^2 = T_p(\mathbb{R}^2\setminus 0)$ be a tangent vector of the form $a\partial_x|_p + b\partial_y|_p$. Compute $i_*(u_*V)$ and $\varphi_*(u_*V)$, where $i : \mathbb{S}^1 \subset \mathbb{R}^2 \setminus 0$ is the standard inclusion map. The answer should be presented in the canonical basis of derivations in $\mathbb{R}$ or $\mathbb{R}^2$.

   (c) (1.5pt) Recall $q : \mathbb{R}^2 \setminus 0 \to \mathbb{RP}^1$ the quotient map. Find a chart $\psi : V \cong \Theta \subset \mathbb{R}$ of $\mathbb{RP}^1$ such that $q(p) \in V$.

   (d) (2pt) Compute $\psi_* q_* V$. What is the relation between $\ker u_*$ and $\ker q_*$?

(4) A subset $C \subset \mathbb{R}^{n+1} \setminus 0$ is a *double cone* if $x \in C, \lambda \in \mathbb{R}^* \implies \lambda x \in C$. As usual, denote $q : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$.

   (a) (1.5pt) Show that for a double cone C, one has $q^{-1}(q(C)) = C$. Conclude that if C is open, then $q(C)$ is open.

   (b) (1pt) Given $e_1 = (1, 0), e_2 = (0, 1)$, find two double cones $e_1 \in C, e_2 \in C'$ that are open and $C \cap C' = \emptyset$

   (c) (2pt) Given arbitrary $v, w \in \mathbb{R}^2 \setminus 0$, $v \neq \lambda w$, find two double cones $v \in C_v, w \in C_w$ that are open and $C_v \cap C_w = \emptyset$.

   (d) (1pt) Prove that $\mathbb{RP}^1$ is Hausdorff.

   (e) (2pt) Generalise the proof above to $\mathbb{RP}^n$.

(1) What is a topological space? A continuous map? A homeomorphism? Give an example of a homeomorphic and a non-homeomorphic pair of spaces, with an explanation.

(2) Let X be a topological space and S a subset. What does it mean for S to be compact (in induced topology)? To be Hausdorff? Show that a compact subset of a Hausdorff space is closed.

(3) What is a topological manifold? Why is $\mathbb{R}^n$ a topological manifold? Are there other examples besides opens in $\mathbb{R}^n$? Name one and present a proof.

(4) What is a presheaf of functions? When is it a sheaf? Give an example of a presheaf that is not a sheaf, and of a presheaf that is a sheaf, with explanations.

(5) What is a smooth (C^∞) manifold? Why is $\mathbb{R}^n$ a smooth manifold? Are there other examples besides opens in $\mathbb{R}^n$? Name one and present its smooth manifold structure (without proof).

(6) What is a smooth map of smooth manifolds? Name one valid criterion as explained in the course. Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth in ordinary sense iff it is smooth as a map of manifolds.

(7) What is a smooth map of smooth manifolds? Name one valid criterion as explained in the course. Give an example of a smooth map that is not a smooth map between opens in Euclidean spaces (without proof). What happens if we compose smooth maps? Justify your response.

(8) Tangent space to a manifold, how is it defined? What happens in the case $M = \mathbb{R}^n$, can we find an explicit basis? Outline the proof of that statement (Taylor formula can be admitted).

(9) Tangent space to a manifold, how is it defined? What is the relation between T_pU and T_pM for $U \subset M$ open (no proof)? What does it imply about the dimension of T_pM?

(10) Given a smooth map $F : M \to N$, how do we define its generalised differential/pushforward $F_* : T_pM \to T_{F(p)}N$? Why is it a linear map? Given smooth maps G, F how does one compute $(G \circ F)_*$ (with proof)?

(11) Given a smooth map $F : M \to N$, how do we define its generalised differential/pushforward $F_* : T_pM \to T_{F(p)}N$? If we consider smooth $F : \mathbb{R}^n \to \mathbb{R}^m$, what is the relation between F_* and $dF(p)$? Provide a justification.

(12) Admitting how tangent spaces and pushforwards work, relate T_pS^n to a subspace in $T_p\mathbb{R}^{n+1}$, justifying your steps.

(13) When is a smooth map a submersion? An immersion? Give example of each. State the canonical submersion/immersion theorem (without proof).

Final exam tips

The materials of the course (everything found on this moodle page) are authorised. I trust you to prepare yourself in advance and not use the internet connection on your machine.

Because of this, it would be in my opinion too revealing to say what sort of a theoretical question will be featured (yes, there will be one). I will point out that since we revised the theory of sections 1-4 in the midterm, the question will be about sections 5-7. It will not ask you to go into abstract proofs, but will rather check your understanding of how certain more "hands-on" notions work and will not deviate from the exposition of the course.

The goal is to make you revise; here are some tips on how to revisit the material of sections 5,6,7 (section 8 is not featured but the orientation part is recommended reading):

(1) Section 5 (vector fields): pretty much everything is useful to understand, except for the integral curves. The proof of relation between vector fields and derivations can be omitted as well. *What is a smooth vector field? Why is it also a derivation of the function algebra? How to pass back and forth between these two descriptions? What does a vector field look like on $\mathbb{R}^n$, or on a chart? What is the Lie bracket of vector fields? How do we pushforward vector fields along a diffeomophism and why does this operation preserve Lie brackets?*

(2) Section 6 (tensors): most of the section, except perhaps the details of the proof of pullback formula for manifolds. *What are k-tensors on a vector space V? What is a tensor product and what are its properties? What happens between k-tensors on V and W given a map $F : V \to W$? How to describe a basis of k-tensors when V is finite dimensional? How do we define tensors on a manifold M, what do they look like when M is $\mathbb{R}^n$ or is a chart? How do we pull back tensors along smooth maps of manifolds?*

(3) Section 7 (forms): it is helpful to study up to and including the proof of existence and uniqueness of de Rham differential on $\mathbb{R}^n$. *What are k-forms on a vector space V? How do we project from k-tensors to k-forms? What is a wedge product and what are its properties? What happens between k-forms on V and W given a map $F : V \to W$? How can one describe a basis of k-forms when V is finite dimensional? How do we define forms on a manifold M, what do they look like when M is $\mathbb{R}^n$ or is a chart? How do we pull back forms along smooth maps of manifolds? What is the de Rham differential, and why does it exist on $\mathbb{R}^n$? How does it interact with pullback along smooth maps $\mathbb{R}^n \to \mathbb{R}^m$?*

In terms of exercises, the following may be found useful (I assume you revised the suggested exercises for the midterm and did your homework):

- PS4: 1,2,4,6,7,8
- PS5: 1,2,3
- PS6: 1,2,3,4 (similar to 3 from PS5) 5 (a-c).

In terms of skills, the exercises may check your knowledge of

(1) what is a topology and how to verify that something is a presheaf or a sheaf,

(2) local form (that is, on opens in $\mathbb{R}^n$) of vector fields, tensor fields and differential forms, how one acts on another.

(3) the operation $f \mapsto df$ and de Rham differential on $\mathbb{R}^n$, the wedge product.

(4) the interpretation of vector fields as derivations of $C^\infty(M)$ and their Lie bracket.

(5) given a smooth map $F : \Omega \to \Theta$ between opens in Euclidean spaces, know how to compute pullbacks of tensors/forms along it.

As you are used with me, the exam will have more problems than you will be able to solve in 2 hours, to offer choice. Even if you do not master some of the topics you should be able to get a decent grade if some other topics you understand well. Good luck.

FINAL EXAM 18/06/2021

Duration: 2 hours. Course materials are allowed. As usual, you can use one question to solve another question, even if you do not know how to do the former. An excellent note is rewarded for 3 complete exercises. Ask questions if you do not understand the formulations!

(1) *(Course question)* Let M be a smooth manifold.

What is a smooth vector field on M? Give any equivalent characterisation. If $M = \mathbb{R}^n$, how can we explicitly describe the smooth vector fields in this case?

How do we define a Lie bracket of two smooth vector fields $X, Y \in \mathcal{T}(M)$? Why is it also a smooth vector field?

(2) *(Topology and sheaves)* We consider $\Omega = \mathbb{R}^2 \setminus 0$. Recall that a subset $C \subset \Omega$ is an *open cone* if C is open in $\mathbb{R}^2$ and $x \in C \implies \lambda x \in C$ for all $x \in C$ and all $\lambda > 0$. Naturally, Ω itself is an open cone, just like the empty set (no elements to verify conditions).

(a) For C_1, C_2 two open cones, show that their intersection is an open cone. For any family $C_i, i \in I$ of open cones, show that their union $\cup_{i \in I} C_i$ is an open cone.

(b) The preceding question shows that $\mathcal{C} = \{C \text{ an open cone}\}$ defines a topology on Ω. Is $(\Omega, \mathcal{C})$ Hausdorff?

(c) Show that the identity map read as $\mathsf{id} : (\Omega, \mathcal{S}) \to (\Omega, \mathcal{C})$ is continuous, where $\mathcal{S}$ denotes the standard topology on Ω (the induced topology for a subset of $\mathbb{R}^2$).
Is the same true about $\mathsf{id} : (\Omega, \mathcal{C}) \to (\Omega, \mathcal{S})$?

(d) We consider $(\Omega, \mathcal{C})$ and define $\mathcal{P}$ by setting

$$\mathcal{P}(C) = \{f : C \to \mathbb{R} \quad | \quad \forall x \in C, \lim_{\lambda \to +\infty} f(\lambda x) = 0\}.$$

Show that $\mathcal{P}$ is a sheaf of functions to $\mathbb{R}$.

(3) *(Explicit computations)* We set $\Omega = \mathbb{R}^2 \setminus 0$ and consider vector fields and forms on it.

(a) Compute the Lie bracket of X, Y where $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.

(b) For X, Y as above, compute $[fX, fY]$ where $f(x, y) = x^2 + y^2$.

(c) For $\eta = \frac{1}{x^2+y^2}(-y\,dx + x\,dy)$, compute $d\eta$ and $d(g\eta)$ where $g(x, y) = (x^2 + y^2)^2$.

(d) Compute $F^*(\eta)$ and $d(F^*(\eta))$ for $F : \mathbb{R}_{>0} \times \mathbb{R} \to \Omega$, $F(r, \theta) = (r\cos\theta, r\sin\theta)$.

(4) *(First-order differential operators)* Let M be a smooth manifold. If needed, without loss of points you can assume $M = \mathbb{R}^n$. For each $f \in C^\infty(M)$, we denote $m_f : C^\infty(M) \to C^\infty(M)$ the multiplication by f: $m_f(h) = f \cdot h$.

(a) Show that m_f is an $\mathbb{R}$-linear map and compute $[m_f, m_g] = m_f \circ m_g - m_g \circ m_f$ for $f, g \in C^\infty(M)$.

(b) Let $X \in \mathcal{T}(M)$ be a smooth vector field on M. Show that

$$[X + m_f, m_g](h) = X(g) \cdot h.$$

In other words, this means that $[X + m_f, m_g] = m_{X(g)}$.

A *first-order differential operator* on M is an $\mathbb{R}$-linear map $L : C^\infty(M) \to C^\infty(M)$ such that for each $g \in C^\infty(M)$, one has $[L, m_g] = m_F$ for some $F \in C^\infty(M)$ (F depends on L and g).

(c) Show that in the definition above, $F = L(g) - g \cdot L(1)$ where $1 \in C^\infty(M)$ is the unity function.

(d) Show that X defined as $X(g) := L(g) - L(1) \cdot g$ is a derivation of $C^\infty(M)$. (hint: try to compute $[L, m_g](f)$ in two different ways).

In particular, any first-order differential operator takes the form $L = X + m_f$, where X is a vector field. One can inductively generalise this notion to define differential operators on M of higher order.

(5) *(Hodge star on $\mathbb{R}^3$)* We work with differential forms on $\mathbb{R}^3 = \{(x, y, z)\}$.

(a) List all standard basis forms in $\Lambda^0(\mathbb{R})$, $\Lambda^1(\mathbb{R})$, $\Lambda^2(\mathbb{R})$, $\Lambda^3(\mathbb{R})$.

(b) For a basis form ω, define $*\omega$ as the unique form such that

$$\omega \wedge *\omega = dx \wedge dy \wedge dz.$$

Find $*\omega$ for all the basis forms that you listed.

(c) We extend $*$ to all forms: if η is a k-form such that $\eta = \sum_i f_i \omega_i$ with ω_i basis k-forms, we put $*\eta := \sum_i f_i * \omega_i$ (for functions, $*f := f * 1$). Compute $* * \eta$ for any k-form η.

(d) For $f \in \Lambda^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3)$, compute $*d * df$.